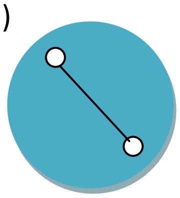
Workshop II: Solving Optimization Problems

For each of the questions below, attempt them yourself and then discuss the questions within your individuals groups. I will ask groups to come to the board and show their solutions.

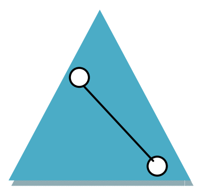
1. A set *U* is said to be *convex* if a line segment containing two points *x* and *y* that lie in the set, also lies within the set, i.e. formally $\ell(x,y) \equiv \left\{ \left(tx + (1-t)y \right) \in U : 0 \le t \le 1 \,|\, x, y \in U \right\}$. [Note that a convex set has nothing to do with a convex function!] For the sets below, indicate whether they are convex or non-convex sets:

   Answer:

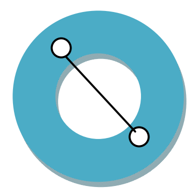
   (a)

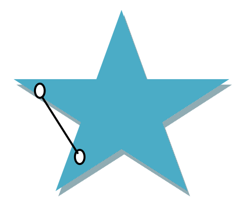
   (b)

   (c)

   (d)

   (a) Convex ; (b) Convex; (c) non-convex; (d) non-convex

2. For the following functions, indicate whether the function is continuous and/or differentiable at the point of transition of its two formulas:

   a. $y = \begin{cases} +x^2, & x \ge 0, \\ -x^2, & x < 0; \end{cases}$

   b. $y = \begin{cases} x^3, & x \le 1, \\ x, & x > 1; \end{cases}$

Answer:

(a) Continuous and differentiable. To see this, examine the derivative of the function in the two regions separately and as they converge to x=0. For the first region, we can see that: $x \ge 0, \lim_{x\to 0} \frac{\partial y}{\partial x} = \lim_{x \to 0} 2x = 0$. *Similarly, for the second region, we can see that:*

$x < 0, \lim_{x\to 0} \frac{\partial y}{\partial x} = \lim_{x\to 0} 2x = 0$. *Hence, since both the left and right hand side derivatives converge to zero as x goes to 0, the function is continuous.*

(b) Continuous but not differentiable. To see this, calculate the derivatives for the two regions and see what happens as x goes to 1. For the first region: $\lim_{x\to 1} \frac{\partial y}{\partial x} = \lim_{x\to 1} 3x^2 = 3$. *For the other region,* $\lim_{x\to 1} \frac{\partial y}{\partial x} = \lim_{x\to 1} 1 = 1$. *Hence since the left and right hand side derivatives do not converge to the same value, and so the function is not differentiable at x=1.*

3. For each of the following functions, defined on $\mathbb{R}^2$, find the stationary points and classify them as local max, local min, saddle point, or "can't tell":
   a. $x^4 + x^2 - 6xy + 3y^2$
   b. $3x^4 + 3x^2 y - y^3$

Answer: Solution strategy involves checking the first and second derivatives with respect to each variable.

(a) The first order conditions are:

$$\frac{\partial}{\partial x} = 4x^3 + 2x - 6y = 0$$
$$\frac{\partial}{\partial y} = -6x + 6y = 0 \Rightarrow x = y$$

The second order conditions are:

$$\frac{\partial^2}{\partial x^2} = 12x^2 + 2; \ \frac{\partial^2}{\partial y^2} = 6 > 0; \frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x} = -6 < 0$$

Hence, possible (x,y) solutions are: (0,0) – saddle point; (1,1) – local min; (-1,-1) – local min. For reasoning why, see answers to second part below.

(b) The first order conditions are:

$$\frac{\partial}{\partial x} = 12x^3 + 6xy = 0 \Rightarrow 6x\left(2x^2 + y\right) = 0 \quad (1)$$

$$\frac{\partial}{\partial y} = 3x^2 - 3y^2 = 0 \Rightarrow x^2 = y^2 \quad (2)$$

From equation (1), we see that a possible solution (Case 1) is x=0, which would imply that y = 0 is also a solution, i.e. (x,y)=(0,0) is a possible solution since it would satisfy equation (2) trivially. Alternatively, $y = -2x^2$ is also a solution (Case 2). To solve for this solution, we can plug $y = -2x^2$ into equation (2), which yields:

$$x^2 = 4x^4$$

$$\Rightarrow 4x^4 - x^2 = 0 \quad = 0$$

$$\Rightarrow x^2\left(4x^2 - 1\right) = 0$$

We have already seen that x = 0 is a possible solution (Case 1), so it represents a repeated solution. The remaining solution is $4x^2 = 1$. *In this solution,* $x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$. *Hence the remaining solutions in Case 2 are,* $(x, y): \left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right)$. *To see how to classify them as max or min, we need to examine the second order conditions. The second order conditions are:*

$$\frac{\partial^2}{\partial x^2} = 36x^2 + 6y \quad (3);$$

$$\frac{\partial^2}{\partial y^2} = -6y \quad (4);$$

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x} = 6x \quad (5);$$

Thus, for case 1, equations (3) – (5) all yield 0 when we plug in (x,y)=(0,0), so we cannot tell if it represents a local maximum or minimum. For Case 2a, $(x, y) = \left(1/2, -1/2\right)$, *equation (3) to (5) yield 6, 3 and 3 respectively, which are all greater than 0. So to determine whether it is a local maximum or minimum, we need to check the determinant of the leading principal minors, which are:*

$36x^2 + 6y$ - first leading principal minor

$\left(36x^2 + 6y\right)\left(-6y\right) - \left(6x\right)^2$ - second leading principal minor

If the leading principal minors alternate in sign, i.e. <0, >0, etc then, $D^2 f$ *, the matrix of second derivatives is negative definite and we have a local maximum. If they are all positive, i.e. >0, >0 etc, then the matrix of second derivatives,* $D^2 f$ *, is positive definite and we have a local minimum. If it were not to follow any of the prescribed patterns above, then the matrix of second derivatives would be said to be indefinite, and this would indicate the presence of a saddle point.*

Hence for case 2a, we can confirm that (1/2,-1/2) is a local minimum since $36x^2 + 6y > 0$ *and* $\left(36x^2 + 6y\right)\left(-6y\right) - \left(6x\right)^2 > 0$.

For the final solution, $(x, y) = \left(-1/2, -1/2\right)$*, equations (3) to (5) yield 6, 3 and -3 respectively. As such, based on the above, the first principal minor is positive, and the second principal minor is also positive, which also indicates the presence of a local maximum.*

4. Find the max and min of $f\left(x, y, z\right) = x + y + z^2$ subject to $x^2 + y^2 + z^2 = 1$ and $y = 0$.

   Answer: This constrained optimization problem can be solved using a Lagrangean by first imposing that y=0 and then constructing the Lagrangean as follows:

$$\mathcal{L} = x + z^2 + \lambda\left[1 - x^2 - z^2\right]$$

   This yields the following first order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda x = 0 \qquad (1)$$
$$\frac{\partial \mathcal{L}}{\partial z} = 2z - 2\lambda z = 0 \quad \Rightarrow 2z\left(1 - \lambda\right) = 0 \ (2)$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - x^2 - z^2 = 0 \ \Rightarrow x^2 + z^2 = 1 \qquad (3)$$

   From equation (2), either $z = 0$ *or* $\lambda = 1$*. For each case, we can find the solutions as follows:*

   Case (1): z=0 :- If $z^* = 0$*, then from equation (3),* $x = \pm\sqrt{1}$*, and thus from equation (1), we obtain* $\lambda^* = \frac{1}{2x^*} = \frac{1}{2}$ *or* $-\frac{1}{2}$.

Case (2): $\lambda^* = 1, x^* = \frac{1}{2}, z^* = \pm\sqrt{\frac{3}{4}}$

5. Consider a firm who uses two inputs to produce a single product. Its production function is given by the Cobb Douglas production function, $Q = x^a y^b$ and if faces output price *p*, and input prices, *w* and *r* respectively. Solve the first order conditions for a profit maximizing input bundle. Use the second order conditions to determine the values of the parameters *a, b, p, w* and *r* for which this solution is a global max.

   Answer: The easiest way to do this is to maximize profits and derive the implications for a, b and the other parameters. As such, profits are:

$$\max_{x,y} \Pi = px^a y^b - wx - ry$$

Differentiating with respect to x and y yields the following first order conditions:

$$\frac{\partial \Pi}{\partial x} = apx^{a-1}y^b - w = 0 \Rightarrow \frac{apQ}{x} = w \Rightarrow x^* = \frac{apQ}{w} \qquad (1)$$

$$\frac{\partial \Pi}{\partial y} = bpx^a y^{b-1} - r = 0 \Rightarrow \frac{bpQ}{y} = r \Rightarrow y^* = \frac{bpQ}{r} \qquad (2)$$

From above, we can see that for a non-trivial solution, i.e. $x \neq 0, y \neq 0$, *the following parameters, a,b,w,r,p>0. Checking the second order conditions leads to the additional implications for the parameters. The second order conditions are:*

$$\frac{\partial^2 \Pi}{\partial x^2} = (a-1)apx^{a-2}y^b = (a-1)a\frac{pQ}{x^2} = (a-1)\frac{w}{x} \qquad (3)$$

$$\frac{\partial^2 \Pi}{\partial y^2} = (b-1)bpx^a y^{b-2} = (b-1)b\frac{pQ}{y^2} = (b-1)\frac{r}{y} \qquad (4)$$

$$\frac{\partial^2 \Pi}{\partial x \partial y} = \frac{\partial^2 \Pi}{\partial y \partial x} = abpx^{a-1}y^{b-1} = ab\frac{pQ}{xy} \qquad (5)$$

For a global max, we need the matrix of second derivatives to alternate in sign. As such, the following conditions need to hold:

$$(a-1)a\frac{pQ}{x^2}=(a-1)\frac{w}{x}<0 \qquad (3') \Rightarrow 0<a<1$$

and

$$\left((a-1)\frac{w}{x}\right)\left((b-1)\frac{r}{y}\right)-\left(ab\frac{pQ}{xy}\right)^2>0$$

$$\Rightarrow (a-1)(b-1)\frac{wr}{xy}-ab\frac{wr}{xy}>0$$

$$\Rightarrow \frac{wr}{xy}[ab-a-b+1]-ab\frac{wr}{xy}>0$$

$$\Rightarrow \frac{wr}{ab}(1-a-b)>0 \qquad (6)$$

Hence from equation (6) above, we see that also 0<b<1. To summarize, for the profit maximization to yield a global maximum, we need:

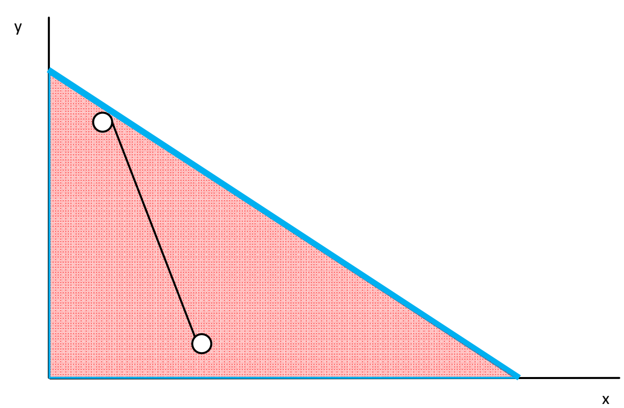
$$a,b\in(0,1);\ p,w,r>0$$

$$and\ x^*=aQ\frac{p}{w};\ y^*=bQ\frac{p}{r}$$

6. Consider the budget constraint $p_x x + p_y y = I$. For $\{p_x, p_y, I, x, y \in \mathbb{R}^+\}$, show that this budget set is a convex set.

   Answer: The easiest way to show this is to graph the budget set. Re-writing the equation above into the equation for a straight line:

$$y=\frac{I}{p_y}-\frac{p_x}{p_y}x$$



Looking at the feasible set above (shaded area), any linear combination of points in the feasible set, also will lie in the set. Hence, the budget set is a convex set.

7. Using a Lagrangean or otherwise, find the general expression (in terms of all the parameters) for the commodity bundle (x_1,x_2) which maximizes the Cobb-Douglas utility function $U\left(x_1,x_2\right)=kx_1^a x_2^{1-a}$ on the budget set $p_1x_1+p_2x_2=I$. Note: $\left\{p_1,p_2,I,x_1,x_2\in\mathbb{R}^+\right\}$ and both *k* and *a* are positive parameters.

Answer: As before, set up the problem as a constrained maximization problem using a Lagrangean as follows:

$$\max_{x_1,x_2} U\left(x_1,x_2\right)=kx_1^a x_2^{1-a} \;\; st: p_1x_1+p_2x_2=I$$

$$\Rightarrow \mathcal{L}=kx_1^a x_2^{1-a}+\lambda\left[I-p_1x_1-p_2x_2\right]$$

$$\Rightarrow FOC:$$

$$\frac{\partial\mathcal{L}}{\partial x_1}=akx_1^{a-1}x_2^{1-a}-\lambda p_1=0\Rightarrow akx_1^{a-1}x_2^{1-a}=\lambda p_1 \quad (1)$$

$$\frac{\partial\mathcal{L}}{\partial x_2}=\left(1-a\right)kx_1^a x_2^{-a}-\lambda p_2=0\Rightarrow\left(1-a\right)kx_1^a x_2^{-a}=\lambda p_2 \quad (2)$$

$$\frac{\partial\mathcal{L}}{\partial\lambda}=I-p_1x_1-p_2x_2=0 \quad (3)$$

We have to solve for x_1, x_2* and λ*. We do this by manipulating the three first order conditions (1) – (3) above to solve for these variables in terms of the parameters and any exogenous variables.*

Let's start with equations (1) and (2) to try and eliminate lambda. We could do this by one of two ways. First, we could re-arrange either of the two equations for lambda and then substitute lambda into the other equation to get x_1 as a function of x_2 (or vice versa). However, a more convenient trick above, since we can see that both equations are linear functions of lambda, is to divide one by the other. This will be slightly easier here (in terms of involving less math) since a lot of the powers of x will cancel out:

$$\frac{akx_1^{a-1}x_2^{1-a}}{(1-a)kx_1^{a}x_2^{-a}} = \frac{\lambda p_1}{\lambda p_2}$$

$$\Rightarrow \frac{ax_2}{(1-a)x_1} = \frac{p_1}{p_2}$$

$$\Rightarrow x_2^* = \frac{(1-a)}{a}\frac{p_1}{p_2}x_1^* \qquad (4)$$

Now, we can substitute equation (4) into equation (3), which we have not used up to this point, to eliminate x_2 as follows:*

$$p_1x_1^* + p_2\frac{(1-a)}{a}\frac{p_1}{p_2}x_1^* = I$$

$$\Rightarrow p_1x_1^*\left[1+\frac{(1-a)}{a}\right] = I$$

$$\Rightarrow \frac{p_1x_1^*}{a} = I$$

$$\Rightarrow x_1^* = \frac{aI}{p_1}$$

Finally, using equation (4), we can recover x_2:*

$$x_2^* = \frac{(1-a)}{a}\frac{p_1}{p_2}x_1^* = \frac{(1-a)}{a}\frac{p_1}{p_2}\left(\frac{aI}{p_1}\right)$$

$$\Rightarrow x_2^* = \frac{(1-a)I}{p_2}$$