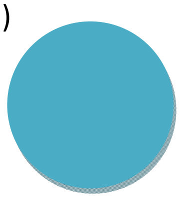
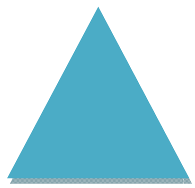
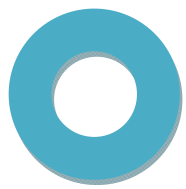
Workshop II: Solving Optimization Problems

For each of the questions below, attempt them yourself and then discuss the questions within your individuals groups. I will ask groups to come to the board and show their solutions.

1. A set *U* is said to be *convex* if a line segment containing two points *x* and *y* that lie in the set, also lies within the set, i.e. formally $\ell(x,y) \equiv \{(tx+(1-t)y) \in U : 0 \le t \le 1 \mid x, y \in U\}$. [Note that a convex set has nothing to do with a convex function!] For the sets below, indicate whether they are convex or non-convex sets:

   (a)  (b)  (c)  (d) 

2. For the following functions, indicate whether the function is continuous and/or differentiable at the point of transition of its two formulas:

   a. $y = \begin{cases} +x^2, & x \ge 0, \\ -x^2, & x < 0; \end{cases}$

   b. $y = \begin{cases} x^3, & x \le 1, \\ x, & x > 1; \end{cases}$

3. For each of the following functions, defined on $\mathbb{R}^2$, find the stationary points, i.e. the particular points where the "slopes" or the first order conditions equal zero.

   a. $x^4 + x^2 - 6xy + 3y^2$

   b. $3x^4 + 3x^2 y - y^3$

4. Find the max and min of $f(x,y,z) = x + y + z^2$ subject to $x^2 + y^2 + z^2 = 1$ and $y = 0$.

5. Consider a firm who uses two inputs to produce a single product. Its production function is given by the Cobb Douglas production function, $Q = x^a y^b$ and if faces output price *p*, and input prices, *w* and *r* respectively. Solve the first order conditions for a profit maximizing input bundle. Use the second order conditions to determine the values of the parameters *a, b, p, w* and *r* for which this solution is a global max.

6. Consider the budget constraint $p_x x + p_y y = I$. For $\{p_x, p_y, I, x, y \in \mathbb{R}^+\}$, show that this budget set is a convex set.

7. Using a Lagrangean or otherwise, find the general expression (in terms of all the parameters) for the commodity bundle (x_1,x_2) which maximizes the Cobb-Douglas utility function $U(x_1, x_2) = kx_1^a x_2^{1-a}$ on the budget set $p_1 x_1 + p_2 x_2 = I$. Note: $\{p_1, p_2, I, x_1, x_2 \in \mathbb{R}^+\}$ and both *k* and *a* are positive parameters.

   [Note (for your own interest): if you managed to show in question 6 that the budget constraint was a convex set, it is also possible to show that the set is both closed and bounded, i.e. compact. Then, there is an actual theorem, the Bolzano-Weierstrass Theorem, that guarantees the existence of a global maximum and minimum for a continuous function on a compact set. Since the Cobb Douglas function above is a continuous function, a solution to the problem above does exist!]