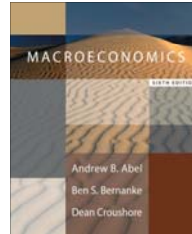


# Business Cycles Econ 402 Professor Yamin Ahmad

## Lecture 2:

- Some tools of Economics
  - Calculus
  - Optimization
  - Regression Analysis



## Some Tools of Analysis

- Part I: Calculus
  - Definitions: Functions, Slopes, Derivatives
- Part II: Indifference Curves and Budget Sets
- Part III: Optimization: Unconstrained and Constrained
- Part IV: Statistics and Regression
  - Descriptive Statistics
  - Understanding and Interpreting Coefficients
  - Some test statistics

Note: These lecture notes are incomplete without having attended lectures

2

## Functions, Slopes and Derivatives

# PART I: CALCULUS

Note: These lecture notes are incomplete without having attended lectures

3

## Functions

- Def: In the statement:  $y = f(x)$ ,  $y \in Y, x \in X$ , the functional notation  $f$  may thus be interpreted as a rule by which the set  $X$  is “mapped” or “transformed” into the set  $Y$ .
  - $x$  is the argument of the function (aka: independent/exogenous variable)
  - $y$  is the value of the function (aka: dependent/endogenous variable)
- Def: The “**Domain**” of a function refers to the set of all permissible values that  $x$  can take in a given context
- Def: The “**Image**” of a function refers to the “ $y$ -value” into which an “ $x$ -value” is mapped
- Def: The “**Range**” refers to the set of all images, which is the set of all values that the  $y$  variable can take.

Note: These lecture notes are incomplete without having attended lectures

4

## Examples

- Linear Functions:  $y = f(x) = mx + c$ . E.g.
  - $y = 2x + 1$
- Polynomial Functions:  $f(x) = ax^b$ 
  - $h(x) = -x^7 + 3x^4 - 10x^2$
- Nonlinear Functions:  $y = f(x)$ . E.g.
  - $y = 1/x$
  - $y = 10^x$

## Question

- For the following functions, report the domain of the functions:

### Answers

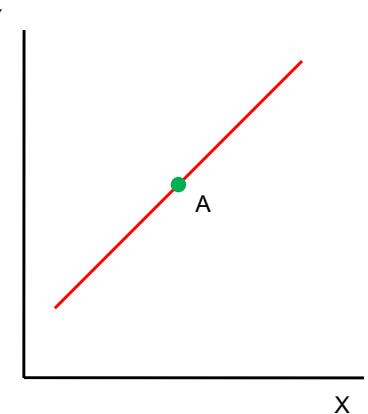
1.  $y = \frac{1}{x-1}$
2.  $y = \frac{1}{\sqrt{x-1}}$
3.  $y = \frac{x}{x^2-1}$
4.  $y = \frac{1}{\sqrt{1-x^2}-1}$

## Concavity and Convexity

- Def: A function is said to be **concave** iff  $\forall x, y \in X$  and  $\forall \alpha \in [0, 1]$ ,
 
$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$$
  - Corollary: for a graph of the function  $f$ , a line segment between two points,  $x$  and  $y$ , always lies below the graph.
- Def: A function is said to be **convex** iff  $\forall x, y \in X$  and  $\forall \alpha \in [0, 1]$ ,
 
$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$
  - Corollary: for a graph of the function  $f$ , a line segment between two points,  $x$  and  $y$ , always lies above the graph.

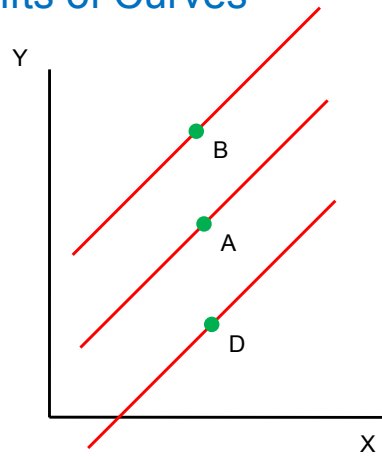
## Movements versus Shifts of Curves

- What causes a movement along a graph, what causes a curve to shift?
- The easiest way to think about it is for a straight line (linear) graph
- Changes in  $Y$  or  $X$  causes a point like  $A$  to move along the curve



## Movements versus Shifts of Curves

- Anything else that affects the relationship between Y and X, e.g. “c” will shift the curve
- Suppose that there is a positive relationship between “c” and Y, then an increase in c will shift the curve up from A to a point like B
- If a negative relationship exists between “c” and Y, an increase in c shifts the curve downwards from A to a point like D

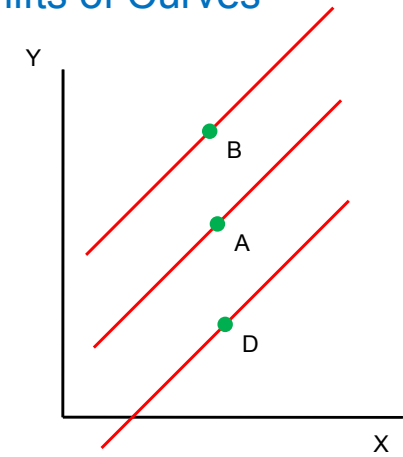


Note: These lecture notes are incomplete without having attended lectures

## Movements versus Shifts of Curves

- Algebraically, for a straight line:  

$$Y = mX + c$$
 where
  - Y is the **dependent variable** (the thing we are looking to explain)
  - X is the **independent variable** (the thing we are using to explain/ relating to Y)
  - m is the **slope**
  - c is the **intercept**
- Thus anything else that impacts Y (apart from X) will change the intercept, c, shifting the curve!



Note: These lecture notes are incomplete without having attended lectures

## The Slope of a Relationship

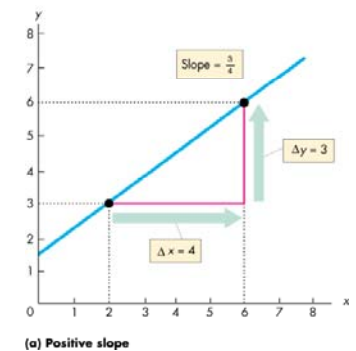
- The **slope** of a relationship is the change in the value of the variable measured on the y-axis divided by the change in the value of the variable measured on the x-axis.
- We use the Greek letter  $\Delta$  (capital delta) to represent “change in.”
- So  $\Delta y$  means the change in the value of the variable measured on the y-axis and  $\Delta x$  means the change in the value of the variable measured on the x-axis.
- The slope of the relationship is  $\Delta y / \Delta x$ .

Note: These lecture notes are incomplete without having attended lectures

## The Slope of a Relationship

### The Slope of a Straight Line

- The slope of a straight line is constant.
- Graphically, the slope is calculated as the “rise” over the “run.”
- The slope is positive if the line is upward sloping.



Note: These lecture notes are incomplete without having attended lectures

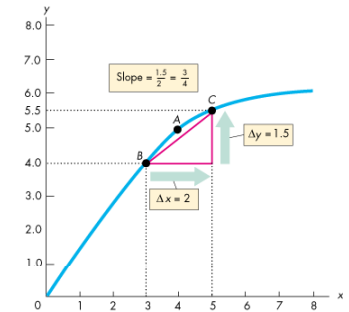
## The Slope of a Relationship

- The Slope of a Curved Line
  - The slope of a curved line at a point varies depending on where along the curve it is calculated.
  - We can calculate the slope of a curved line either at a point or across an arc.

## The Slope of a Relationship

### Slope Across an Arc

- The *average* slope of a curved line across an arc is equal to the slope of a straight line that joins the endpoints of the arc.
- Here, we calculate the average slope of the curve along the arc BC.

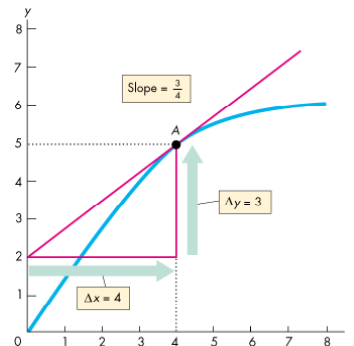


Quick question: Is the function above concave or convex?

## The Slope of a Relationship

### Slope at a Point

- The slope of a curved line at a point is equal to the slope of a straight line that is the tangent to that point.
- Here, we calculate the slope of the curve at point A.



## More on Calculating Slopes

### Slope of a Linear Function

- Consider a linear function:
 
$$y = f(x) = mx + c$$
- Suppose that the pairs  $(x_0, y_0)$  and  $(x_1, y_1)$  both represent arbitrary points on the line. Then the ratio:
 
$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$
 is called the slope.
- $c$  is called the intercept
- Question: What is the slope of a line that goes through the points  $(4, 6)$  and  $(0, 7)$ ?

## Solution...

- Take the points (4,6) and (0,7)
- If we plug into the formula on the previous page:

$$\frac{\Delta y}{\Delta x} = m = \frac{y_1 - y_0}{x_1 - x_0}$$

$$= \frac{7 - 6}{0 - 4} = \frac{1}{(-4)} = -\frac{1}{4}$$

- Hence the slope of a line that goes through the points (4,6) and (0,7) is -1/4.

## Another Practice Problem... (Solution next page)

### Centigrade and Fahrenheit

- Let C denote the temperature in degrees Centigrade and let F denote the temperature in degrees Fahrenheit.
- We know that 0° Centigrade is 32° Fahrenheit is the freezing temperature of water, and that 100° Centigrade or 212° Fahrenheit is the boiling point for water.
- What is the equation that relates degrees Fahrenheit to degrees Centigrade? What is the inverse function, i.e. the equation that relates degrees Centigrade to degrees Fahrenheit?

## Practice Problem... Answer

### Centigrade and Fahrenheit

- To find the equation of the line through points (0,32) and (100,212), we first find the slope:

$$\text{slope, } m = \frac{\Delta F}{\Delta C} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}$$

- This means an increase of 1° Centigrade corresponds to an increase of 9/5° Fahrenheit. Hence we can use the slope, 9/5 and the point (0,32) to express the linear relationship:

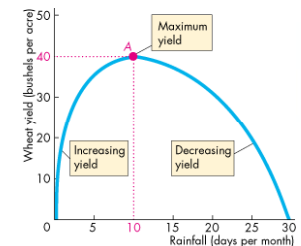
$$\frac{F - 32}{C - 0} = \frac{9}{5} \quad \text{or} \quad F = \frac{9}{5}C + 32$$

- Or equivalently (re-arranging for C=f(F))

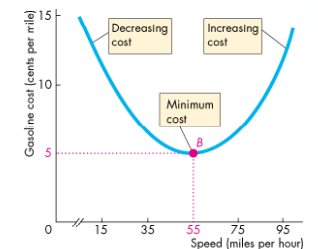
$$9C = 5(F - 32) \quad \text{i.e.} \quad C = \frac{5}{9}(F - 32)$$

## How to find Extremals ...?

- Take our two graphs from earlier...
- ... how would we find any minima or maxima that exist?
- Answer:
  - Look for “stationary points”, i.e. where the slopes of the curves are zero!



(a) Relationship with a maximum



(b) Relationship with a minimum

## How to find Extremals ...?

- To find any maxima or minima, use the following methodology:
  - Calculate the slope of the function  $y=f(x)$ , i.e.  $\Delta y/\Delta x$
  - Set  $\Delta y/\Delta x=0$  and find values of  $x$  where the slope equals zero!
  - Check around those points found in step 2.
    - If the slope is increasing then decreasing, it is a maximum
    - If the slope is decreasing then increasing, then it is a minimum.

Note: These lecture notes are incomplete without having attended lectures

21

## Slope Terminology

- For a linear function:  $y = f(x) = mx + c$ , the slope is easy to calculate:

$$\text{slope} = \frac{\Delta y}{\Delta x} = m$$

- When the change,  $\Delta$ , is very (very, very, very, ...) small, i.e. as  $\Delta \rightarrow 0$ , we write the change in  $y$  relative to  $x$  as:

$$\frac{dy}{dx}$$

and call it the “*derivative*” of  $y$  with respect to  $x$

Note: These lecture notes are incomplete without having attended lectures

22

## Formal Definition of a Derivative

- Def: Let  $f$  be defined (and real valued) on  $[a,b]$ . For any  $x \in [a,b]$  form the quotient:

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, t \neq x)$$

and define  $f'(x) = \lim_{t \rightarrow x} \phi(t)$  provided that this limit exists.

- This is called the derivative of  $f$ .
- If  $f'(\cdot)$  is defined at a point, then we say that  $f$  is **differentiable at  $x$** .
- If  $f'(\cdot)$  is defined at every point on  $[a,b]$ , we say **that  $f$  is differentiable on the interval  $[a,b]$** .

Note: These lecture notes are incomplete without having attended lectures

23

## Some Rules to Calculate Slopes

For **polynomial functions**, use the following rules:

- The slope of a constant = 0. e.g. slope of  $y=5$  is 0.
- The slope of a variable to the  $k^{\text{th}}$  power, e.g.  $y=x^k$  is:

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = kx^{k-1},$$

For example: the slope of  $y=x^4$  is:  $\frac{dy}{dx} = 4x^3$

- $y=Ax^k$  is  $\frac{dy}{dx} = Akx^{k-1}$ , for example: for  $y=5x^4$ , the slope is:

$$\frac{\Delta y}{\Delta x} = 5 * 4x^3 = 20x^3$$

Note: These lecture notes are incomplete without having attended lectures

24

## Some Rules to Calculate Slopes (cont.)

The slope of:

4. Combining (3) and (1), we get:

i. For  $y = Ax^k + c$ ,  $\frac{\Delta y}{\Delta x} = Akx^{k-1}$ ,

e.g. for  $y=5x^4 + 10$ ,  $\frac{\Delta y}{\Delta x} = 5 * 4x^3 = 20x^3$

ii. For  $y = Ax^k + Bx^n$ ,  $\frac{\Delta y}{\Delta x} = Akx^{k-1} + Bnx^{n-1}$ ,

e.g. for  $y=5x^4 + 3x^5$

$$\frac{\Delta y}{\Delta x} = 5 * 4x^3 + 3 * 5x^4 = 20x^3 + 15x^4$$

## Partial Derivatives

- For functions of several variables, we can calculate the **partial derivative** with respect to a certain variable by treating the remaining variables as constants and differentiating as usual by using the rules of one-variable calculus.

- If  $z = f(x,y)$  is a function of two variables, the two partial derivatives are denoted :

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

$$\text{or } \frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}; \text{ or } f_x \text{ and } f_y$$

## Some Rules to Calculate Slopes (cont.)

The slope of:

5. If  $y$  is a function of several variables, e.g.  $y=f(x,s,t)$ , treat the other variables as **constant** when calculating the slope of  $y$  w.r.t.  $x$ , i.e. the partial derivative of  $y$  w.r.t.  $x$ :

e.g.  $y=100 + 5x^4 + 10s^2 + 15t^3$

$$\frac{\partial y}{\partial x} = 5 * 4x^3 = 20x^3;$$

Similarly  $\frac{\partial y}{\partial s} = 10 * 2s = 20s;$

$$\frac{\partial y}{\partial t} = 15 * 3t = 45t$$

## Summary of Single Variable Calculus Rules

Quick Summary :

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$x^m$	$mx^{m-1}$
$Ae^{bx}$	$Abe^{bx}$
$\ln x$	$(1/x)$
$a^x$	$a^x \ln a$

## Practice Questions I:

Calculate the slopes for the following functions

Calculate the derivative of y with respect to each variable:

1.  $y = x^3 - 12x^2 + 36x + 8$
2.  $y = x^2 + 2xt + t^2$
3.  $y = x^2t + t^2x - 2xt + 5x - 3t$
4.  $y = 10K^\alpha L^{(1-\alpha)}$
5.  $y = 4e^{2x}$
6.  $y = 5\ln x + 10^x$

Quick Summary :

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$x^m$	$mx^{m-1}$
$Ae^{bx}$	$Abe^{bx}$
$\ln x$	$(1/x)$
$a^x$	$a^x \ln a$

## Some Rules to Calculate Slopes (cont.)

6. **The Chain Rule:** In single variable calculus, the chain rule is quite simple and looks as follows. Suppose that  $y=f(u)$  and  $u=g(x)$ . Then we can differentiate y with respect to x using the chain rule as follows:

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

e.g.  $y=10u^2$ ;  $u=2x+4$ ; Then

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} = 20u \cdot 2 = 40(2x+4) = 80x + 160$$

The multivariate version of the chain rule follows a similar logic.

## Some Rules to Calculate Slopes (cont.)

7. **The Total Derivative:** The total derivative is obtained by attributing any change in an endogenous variable to individual changes in exogenous variables, treating the remaining exogenous variables as constants.

- Another way to think about it, is that it is a Taylor Expansion of a function about a particular point:

e.g.  $z = f(u, v) = 2uv^2$

Then  $dz = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$

$$\Rightarrow dz = 2v^2 du + 4uv dv$$

## Some Rules to Calculate Slopes (cont.)

8. **The Product Rule:** Suppose that  $z=xy$ . Then we can differentiate z with respect to x using the product rule as follows:

$$dz = ydx + xdy$$

- Note: This is simply the total derivative of the function z!
- E.g. Labor income = Wages x Labor

$$\therefore \Delta \text{Labor Income} = \text{Labor} \times \Delta \text{Wages} + \text{Wages} \times \Delta \text{Labor}$$

## Some Rules to Calculate Slopes (cont.)

9. **The Quotient Rule:** Suppose that  $z = \frac{x}{y}$ .

Then we can differentiate  $z$  with respect to  $x$  using the quotient rule as follows:

$$dz = \frac{ydx - xdy}{y^2} = \frac{dx}{y} - \left(\frac{x}{y}\right) \frac{dy}{y}$$

- Note: Once again, this is simply the total derivative of the function  $z$ !

Note: This is just simply:

$$GDP\ Deflator \times \frac{\Delta RGDP}{RGDP}$$

- E.g.  $GDP\ Deflator = \frac{NGDP}{RGDP}$

$$\therefore \Delta GDP\ Deflator = \frac{1}{RGDP} \times \Delta NGDP - \frac{NGDP}{RGDP^2} \Delta RGDP$$

## Higher Order Derivatives

- If all the partial derivatives exist for a function, and the partial derivatives themselves are differentiable, we can then take the derivative of a derivative (and so on...).
- Example:

$$z = f(x) = 4x^3y^2$$

$$\frac{\partial z}{\partial x} = 12x^2y^2; \frac{\partial z}{\partial y} = 8x^3y$$

$$\frac{\partial^2 z}{\partial x^2} = 24xy^2; \frac{\partial^2 z}{\partial y^2} = 8x^3; \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 24x^2y$$

Young's Theorem

# PART II: INDIFFERENCE CURVES AND BUDGET SETS

## Indifference Curve Analysis

**Tool of Analysis:** Indifference Curves

- Represents demand side of the economy (consumers)
- **Indifference Curve** — shows combinations of two goods that yield the same level of satisfaction (“utility”) to a consumer.

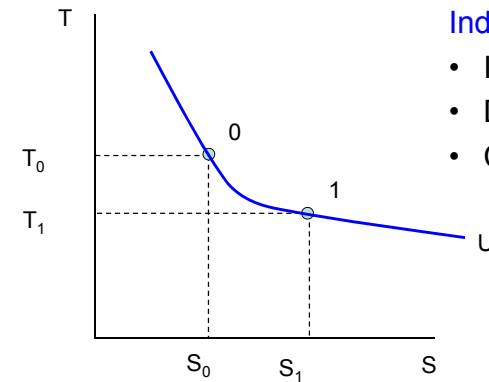
## A Quick Question For You

Think of two goods/items that you like. Let's call them good S and good T. Suppose that you have 10 of each type of good. Now consider the following:

In a table, write down numbers for:

- **Additional S:** How many units of good T would you be willing to give up for an additional unit of good S? ... or for the 12<sup>th</sup>, 13<sup>th</sup> or 14<sup>th</sup> unit of good S?
- **Additional T:** How many units of good S would you be willing to give up for an additional unit of good T? ... or for the 12<sup>th</sup>, 13<sup>th</sup> or 14<sup>th</sup> unit of good T?
- **Plot these on a graph! What shape have you drawn?**

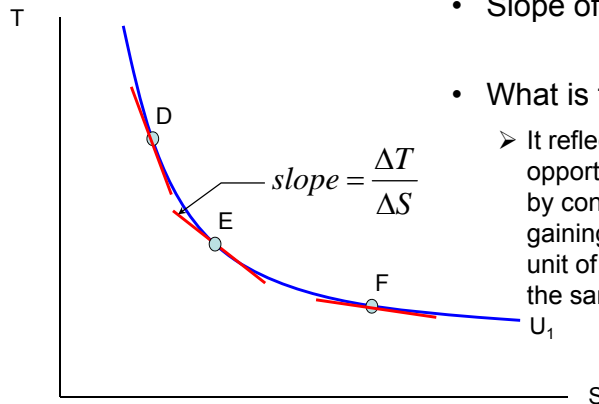
## An Indifference Curve



Indifference Curves are:

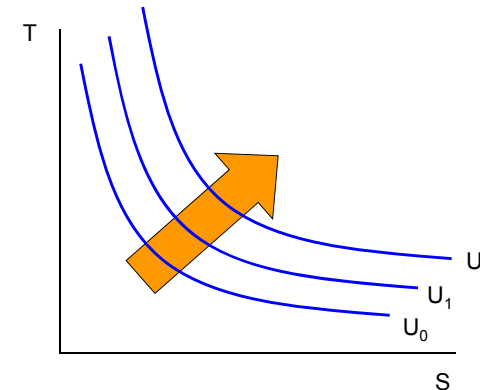
- Individual Specific
- Downward Sloping
- Convex to the origin

## Marginal Rate of Substitution (MRS)



- Slope of IC = -MRS  
=  $\Delta T / \Delta S$
- What is the MRS?
  - It reflects the opportunity cost faced by consumers of gaining an additional unit of good S, but at the same level of utility.

## Indifference Curves (cont.)



- Higher Indifference Curves represent higher levels of utility.
- Why?  $U_1$  and  $U_2$  represent combinations of T and S that are at least the same (if not more) of either good (compared to  $U_0$ ).

## Some Things to Think About...

- Question: Can indifference curves cross?
- Answer:
  
- Question: Are the indifference curves “parallel”?
- Answer:

Note: These lecture notes are incomplete without having attended lectures

41

## Properties of Indifference Curves

To summarize, indifference curves are:

- Individual-specific
- Downward-sloping
- Convex to the origin
- Higher curves indicate higher levels of satisfaction
- Non-intersecting
- Slope of indifference curve is the **marginal rate of substitution (MRS)**

Note: These lecture notes are incomplete without having attended lectures

42

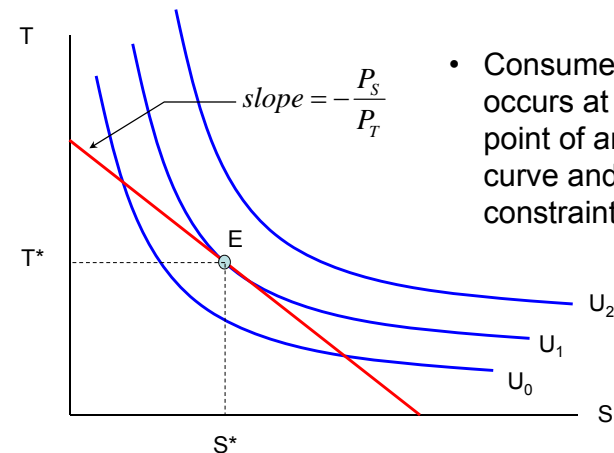
## Example: Consumer Utility Maximization

- **Consumer maximizes utility** subject to an income or **budget constraint**
- What does this mean?...
  - Given your budget (income), you try and pick combinations of S and T that lie within your budget whilst giving you the greatest utility!
- Question: Suppose that you have an income, Y, which you want to spend on two goods, S and T, which cost  $P_S$  and  $P_T$ .
  - Write out the budget constraint!

Note: These lecture notes are incomplete without having attended lectures

43

## Consumer Utility Maximization



- Consumer solution occurs at the tangency point of an indifference curve and the budget constraint.

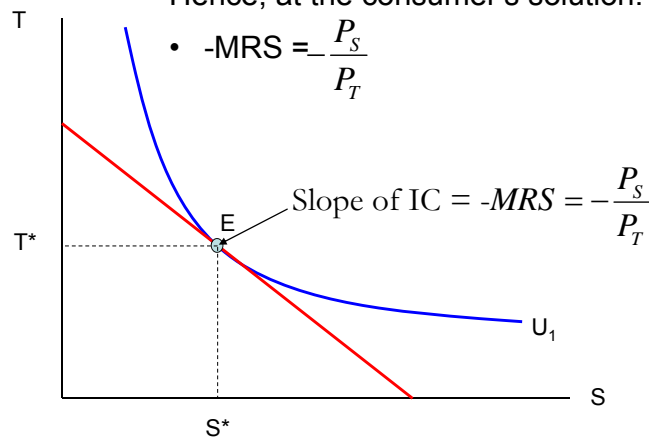
Note: These lecture notes are incomplete without having attended lectures

44

## Consumer Utility Maximization

Hence, at the consumer's solution:

- $-MRS = -\frac{P_S}{P_T}$



Note: These lecture notes are incomplete without having attended lectures

45

Lagrange's Theorem and Kuhn-Tucker Conditions

## PART III: OPTIMIZATION

Note: These lecture notes are incomplete without having attended lectures

46

## Optimization

- Optimization problems typically involve trying to find an extremal point for a particular function, i.e. a maximum or minimum.
- As such, the optimization may be:
  - unconstrained, e.g.  $\max y = -2(x-5)^2$
  - constrained where the function you are trying to maximize is subject to one or more constraints, e.g. the utility maximization problem from Part II.

Note: These lecture notes are incomplete without having attended lectures

47

## Unconstrained Optimization

- Suppose that we are interested in maximizing a function,  $f(x)$ , i.e.  $\max_x f(x)$
- This means that if a point  $x^*$  is a maximum (minimum), then the image of all points around  $x^*$  should be smaller than the image of  $x^*$ , i.e.
  - Formally, Define: a point  $x^* \in \text{int}(X)$  is a **relative (local) maximum** of  $f$  iff  $\exists r > 0: \forall x \in N_r(x^*), f(x^*) \geq f(x)$
- Graphically this means .....

Note: These lecture notes are incomplete without having attended lectures

48

## Necessary Conditions for an optimum

- Recall from earlier: to find a maximum or minimum *at an interior solution*, we said that we should look for when the slope of the function is zero, i.e.
  - We look at the **first order condition** by differentiating the function and setting the derivative equal to zero!
  - Example:  $y = -2(x - 4)^2$ 

$$\frac{dy}{dx} = -4(x^* - 4) = 0 \Rightarrow x^* = 4$$
- However this provides us with a **necessary condition**.
  - A condition that has to hold true at the optimum, but need not be sufficient for a maximum or minimum.

Note: These lecture notes are incomplete without having attended lectures

49

## Sufficient Conditions for an optimum

- To find a **sufficient condition** for a maximum (minimum) we need to look at the **second order conditions** by examining the second derivative.
- If indeed we have a true local maximum (minimum), then all the other images of points around  $x^*$  should be less (greater) than the image of  $x^*$ .
- This can only occur if the second derivative is negative (positive) or zero, i.e. negative (positive) semi-definite.
- E.g. for the function on the previous slide,  $\frac{d^2y}{dx^2} = -4 < 0$

Note: These lecture notes are incomplete without having attended lectures

50

## Sufficient Conditions for an optimum

Hence if the second derivative is:

- Negative, then  $x^*$  is a strict local maximum
- Positive, then  $x^*$  is a strict local minimum
- Indefinite, then  $x^*$  is neither a local maximum or minimum. In this case, we say that  $x^*$  is a saddle point,
  - i.e.  $x^*$  maximizes the function in some directions and minimizes the function in other directions.

Note: These lecture notes are incomplete without having attended lectures

51

## Question: A Discriminating Monopolist

- Consider a monopolist who is able to price discriminate between two types of consumers.
- The monopolist charges a price:
  - $P_1 = 50 - 5Q_1$  for Type 1 consumer
  - $P_2 = 100 - 10Q_2$  for the Type 2 consumer
- The cost function for the monopolist is captured by:
  - $C = 90 + 20Q$ , where  $Q$  represents the total quantity produced
- How much should the monopolist produce for each type of consumer if he wants to maximize profits? How much should he charge each type of consumer?

Note: These lecture notes are incomplete without having attended lectures

52

## Constrained Optimization

- Most economic problems try to maximize or minimize a function relative to a set of constraints, e.g. maximize utility subject to a budget constraint.
- Example:  

$$\max_{x,y} f(x,y) \text{ subject to } g(x,y) \leq b$$
- One possible way to solve this would be to internalize the constraint into the objective function and solve for the maximum.
- The other is to construct and use a Lagrangean function.

Note: These lecture notes are incomplete without having attended lectures

53

## Using a Lagrangean Function

- Construct the Lagrangean function as:

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda(b - g(x, y))$$

where  $\lambda$  is known as the **Lagrange Multiplier**.

- In order to find the maximum, differentiate the Lagrangean and evaluate the following first order conditions (FOC):
  - $\frac{\partial \mathcal{L}}{\partial x}(x^*, y^*, \lambda^*) = 0$  ;  $\frac{\partial \mathcal{L}}{\partial y}(x^*, y^*, \lambda^*) = 0$
  - $\lambda^* [b - g(x^*, y^*)] = 0$
  - $\lambda^* \geq 0$
  - $g(x^*, y^*) \leq b$

Note: These lecture notes are incomplete without having attended lectures

54

## Example

- Suppose that a consumer consumes goods  $x_i$ ,  $i=1,2$  and has a utility function:  $U(x_1, x_2) = x_1 x_2$ . In addition they face the budget constraint we saw earlier:  $p_1 x_1 + p_2 x_2 = I$
- We can construct the Lagrangean as follows:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 x_2 + \lambda(I - p_1 x_1 - p_2 x_2)$$

- The first order conditions are:
 
$$x_2 - \lambda p_1 = 0$$

$$x_1 - \lambda p_2 = 0$$

$$\lambda[I - p_1 x_1 - p_2 x_2] = 0$$

Note: These lecture notes are incomplete without having attended lectures

55

## Example continued:

- Notice that if  $\lambda = 0$ , then both  $x_1 = x_2 = 0$ , which is the trivial solution.
- So, by eliminating  $\lambda$  from the first two first order conditions, we get:

$$\lambda = \frac{x_1}{p_2} = \frac{x_2}{p_1} \Rightarrow x_1 = \frac{p_2 x_2}{p_1}$$

- Plugging into the third equation yields:

$$x_1^* = \frac{I}{2p_1}, x_2^* = \frac{I}{2p_2}, \lambda^* = \frac{I}{2p_1 p_2}$$

- Note: the second order conditions are trivially satisfied.

Note: These lecture notes are incomplete without having attended lectures

56

## Summary on Constructing the Lagrangean Function

- Prior to correctly setting up a constrained optimization problem, you need to do the following:
  - Identify the objective** of the economic agent – what is the objective that the agent is trying to achieve, e.g. maximizing profits, maximizing utility, minimizing costs, etc.
  - Identify the constraints** faced by the economic agent, e.g. is the individual subject to a budget constraint, or resource constraint.
  - Identify the “choice” (or “control”) variables** – i.e. the particular variables the economic agent actually chooses or decides upon, e.g. consumption/savings, inputs for firms.

Note: These lecture notes are incomplete without having attended lectures

57

## Summary on Constructing the Lagrangean Function

- The following rules summarize how to construct the Lagrangean function for the different scenarios:
 

<u>Objective:</u>	<u>Correct form for Lagrangean:</u>
1. Maximize $f(.)$ st: $g(x,y) \leq b$ :	$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda [b - g(x, y)]$
2. Minimize $f(.)$ st: $g(x,y) \geq b$ :	$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda [b - g(x, y)]$
3. Maximize $f(.)$ st: $g(x,y) \geq b$ :	$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda [b - g(x, y)]$
4. Minimize $f(.)$ st: $g(x,y) \leq b$ :	$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda [b - g(x, y)]$

Note: These lecture notes are incomplete without having attended lectures

58

## Practice Question

- Solve the following maximization problem for  $x$  and  $y$ :

$$\max f(x, y) = xy \quad \text{st: } g(x, y) = x^2 + y^2 \leq 1$$

- Answer:

Note: These lecture notes are incomplete without having attended lectures

59

## Multiple Constraints

- The basic intuition behind constructing the Lagrangean follows through when there are multiple constraints.
- For each constraint, include the constraint in the Lagrangean and multiply the constraint by the Lagrange multiplier based on whether the objective function is being maximized or minimized, and whether the constraint is a greater than or less than constraint.

Note: These lecture notes are incomplete without having attended lectures

60

## What are the Lagrange Multipliers?

- The Lagrange multipliers play an important role in economic analysis and often have an important economic meaning.
- The multipliers measure the extent to which the optimal value of the objective function changes as a result of changing (or relaxing) the constraint by one unit.
- As such, they provide a natural measure of **value** for scarce resources in optimization problems:-
  - They are often referred to as the **shadow price** for a resource

Note: These lecture notes are incomplete without having attended lectures

61

## PART IV: STATISTICS AND REGRESSION

Note: These lecture notes are incomplete without having attended lectures

62

## Now for some Statistics

### Key Concepts:

- Distributions
- Moments
- Random Variables
- Population versus Sampling
  - Descriptive Statistics
  - Correlation

Note: These lecture notes are incomplete without having attended lectures

63

## Descriptive Statistics

- Averages:
  - Mean
  - Median
  - Mode
- Measures of Dispersion:
  - Variance
  - Standard Deviation
- Covariance and Correlation between different variables (or same variable across different points in time!)

Note: These lecture notes are incomplete without having attended lectures

64

## Averages from Data

Suppose you have some data on a particular random variable,  $x$ , i.e.  $x_i = \{x_1, x_2, x_3, \dots, x_n\}$ ,  $i = 1, \dots, n$ , are the  $n$  observations forming your random sample

*E.g. height of people in this room at the very moment.*

### Different Measures of Averages

- Mean:  $\mu = E[x] = \frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{n} (x_1 + x_2 + \dots + x_n)$
- Weighted Mean:  $\bar{x} = \frac{1}{n} \sum_{i=1}^n w_i x_i = \frac{1}{n} (w_1 x_1 + w_2 x_2 + \dots + w_n x_n)$

where  $w_i$  are the weights on each observation,  $i = 1, \dots, n$

## Other Measures of Central Tendencies

- Median: is the value  $m$ , such that:
  - $\text{Prob}(X \leq m) \geq \frac{1}{2}$
  - And  $\text{Prob}(X \geq m) \leq \frac{1}{2}$

in other words: the median value in a data sample for a random variable, is the middle ranked observation, i.e. the particular observation where at least 50% of all other observations are smaller than it.

- Mode: is the value of  $x_i$  occurring most frequently in the sample.

## Standard Deviation and Variance

- The variance is a measure of dispersion in the sample.

$$\begin{aligned} \text{Var}(x) &= E[(x - \mu)^2] = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu)^2 \\ &= \frac{1}{n-1} \left[ \sum_{i=1}^n x_i^2 - n\mu^2 \right] \end{aligned}$$

- Variance of  $x$  is represented by the symbol  $\sigma_x^2$  for the population, and  $s_x^2$  for a sample.
- Standard Deviation is the (positive) square root of the variance, and is represented by symbol  $\sigma_x$  for the population, and  $s_x$  for a sample.

## Covariance and Correlation

- For a sample of 2 or more variables, we can compute the covariance and correlation between different variables.
- We can also compute the covariance or correlation of a single variable over time
- $\text{Cov}(x, y) = E[(x - \mu_x)(y - \mu_y)] = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) = s_{xy}$
- Correlation for a population,  $\text{Corr}(x, y) = \rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$

## Covariance and Correlation

- Covariance for a population equals

$$\text{Cov}(x, y) = E[(x - \mu_x)(y - \mu_y)] = \sigma_{xy}$$

- The sample counterpart equals:

$$\text{Cov}(x, y) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) = s_{xy}$$

- Similarly, the population (sample) correlation for two variables (x,y) is defined as the covariance divided by the (sample) standard deviation of each variable:

$$\text{Corr}(x, y) = \rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} \quad (\text{sample: } r_{xy} = \frac{s_{xy}}{s_x s_y})$$

## Practice Question:

For the table to the right, calculate the following:

Observation	Income I	Education E	Observation	Income I	Education E
1	20.5	12	11	55.8	16
2	31.5	16	12	25.2	20
3	47.7	18	13	29.0	12
4	26.2	16	14	85.8	16
5	44.0	12	15	15.1	10
6	8.28	12	16	28.5	18
7	30.8	16	17	21.5	16
8	17.2	12	18	17.7	20
9	19.9	10	19	6.42	12
10	9.96	12	20	84.9	16

- Means
- Standard Deviations
- Covariance
- Correlation

Note: Income in Thousands; Education in Years

## Regression Analysis: Analyzing Theory

- Economic Theory** specifies a set of relationships between variables.
  - E.g. Demand equations, production functions, consumption function, etc.
- Econometrics**: Empirically investigate economic theory using data to provide estimates of key (unknown) parameters in economic models
  - E.g. estimates of elasticities, marginal propensity to consume, etc.

## An Example: Consumption Function

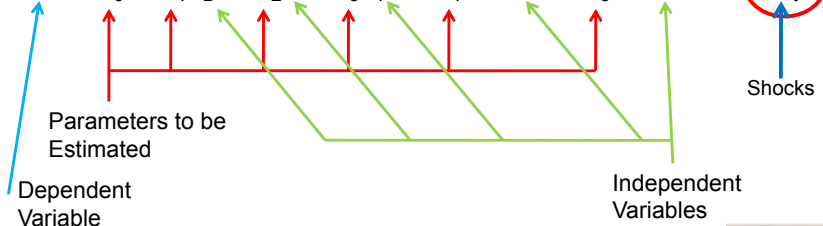
- Consider a household's consumption function.
- Step 1: Posit what kind of factors might influence household spending behavior?
  - Choose factors that you think might impact consumption when they change
- Possible Factors:-
  - Disposable Income
  - Real Interest rate
  - Expected future income
  - Stock Market
  - Weather
  - ...etc...

## An Example: Consumption Function

- How might we examine the effects of these different factors on consumption?
  - Step 2: Set consumption as a function of these variables, i.e. **regress** consumption on a set of explanatory variables

Thus the consumption function might look like:

$$C = c_0 + c_1 Y_D + c_2 r + c_3 Y_f^e + c_4 Stock + c_5 Weather + \varepsilon_t$$



Note: These lecture notes are incomplete without having attended lectures

## An Example: Consumption Function

- Step 3: Make some assumptions about the variables and the disturbance/shock terms: e.g.
  - Linear relationship between dependent and independent variables and shock term
  - Shocks :
    - are uncorrelated with explanatory variables
    - have Finite moments
    - are spherically distributed
    - ... are perhaps even Normally distributed
  - Etc...
- Step 4: Estimate coefficients etc, using data and least squares methodology (minimize sum of squared residuals)

Note: These lecture notes are incomplete without having attended lectures

## Estimation Results

- Here are some results from such an estimation.

Dependent Variable: RCONS  
 Method: Least Squares  
 Date: 09/02/08 Time: 13:45  
 Sample (adjusted): 1965Q1 2000Q4  
 Included observations: 144 after adjustments

	Coefficient	Std. Error	t-Statistic	Prob.
Constant	-62818.85	3029.03	-20.74	0.00
Real GDP	0.24	0.02	11.75	0.00
Real Fed Funds Rate	1.24	0.75	1.66	0.08
Expected Future Income	7452.52	4839.30	1.54	0.12
Stock Price index	2.59	0.47	5.48	0.00
Weather Index	-3.32	2.92	-1.14	0.26

- What do the numbers mean?

- Things to look for:
  - Coefficients
  - Standard Errors
  - Significance of t-Statistics or P-value
  - R-Squared ("Goodness of Fit")
  - Significance of F-Statistic

R-squared	0.999255	Mean dependent var	13673.72
Adjusted R-squared	0.999234	S.D. dependent var	2835.059
S.E. of regression	78.48636	Akaike info criterion	11.59783
Sum squared resid	856255.2	Schwarz criterion	11.70095
Log likelihood	-830.0439	Hannan-Quinn criter.	11.63973
F-statistic	46610.98	Durbin-Watson stat	0.523971
Prob(F-statistic)	0		

Note: These lecture notes are incomplete without having attended lectures

## Practice Question:

- Look at the estimation results to the right.
- Which variables are "significant"?
  - Pick one coefficient and interpret it.
- How good was the estimation?

Dependent Variable: Real Exchange Rate,  $y_t$

Variable	US-CAN	US-FRA	US-ITA	US-JAP	US-UK
# lags in equation	2	4	2	2	3
Constant	-0.002 (-1.532)	0.002 (1.267)	0.000 (0.244)	0.007 (2.175)	0.004 (2.169)
$y_{t-1}$	1.193 (24.011)	1.283 (22.623)	1.321 (24.645)	1.283 (26.698)	1.343 (26.579)
$y_{t-2}$	-0.200 (-4.041)	-0.401 (-4.368)	-0.342 (-6.376)	-0.298 (-6.236)	-0.484 (-5.946)
$y_{t-3}$		0.250 (2.710)			0.118 (2.345)
$y_{t-4}$		-0.154 (-2.729)			
N	394	312	312	394	394
R <sup>2</sup>	0.991	0.972	0.973	0.983	0.967
Sum squared resid	0.056	0.196	0.181	0.299	0.227
Standard error	0.0120	0.0254	0.0243	0.0277	0.0242
Q(p)	0.170	1.140	0.546	1.987	3.097
Durbin Watson	1.989	1.989	1.970	1.976	1.977

Note: The numbers in parenthesis are t-ratios. Q(p) is the Ljung-Box (1979) statistic that is

repeated here from table (2) and is distributed as a  $\chi^2_{(p)}$  statistic. The critical values for the

$\chi^2_{(2)}$  is 5.99; for a  $\chi^2_{(3)}$  is 7.81, for a  $\chi^2_{(4)}$  is 9.49

Source: Yamin Ahmad and Stuart Glosser, 2009, "Searching for Nonlinearities in Real Exchange Rates", Applied Economics, Table 2,

Note: These lecture notes are incomplete without having attended lectures

## The End

### Things to take away:

- Movement versus shifts of curves
- How to calculate slopes
  - Derivative and how to calculate it
- Indifference Curves and Budget Sets
- What optimization involves
- Calculating Descriptive Statistics
- Understanding what a regression is and components of a regression
- Examining regression results and determining which variables are important