Sample Midterm Solutions

Instructions: Please answer both questions. You should show your working and calculations for each applicable problem. Correct answers without working will get you relatively few points. Make sure to put your name and id number on the top of every single piece of paper that you turn in. Also, please write in a page number at the bottom of every piece of paper that you turn in and assemble them sequentially before submitting.

Question 1: Math and Optimization (50%)

Consider the "Primal" problem faced by a producer. Namely, a producer (or firm) can be thought to maximize profits, subject to producing their goods based on a particular kind of technology, or production function.

Suppose that production of good *y* requires two inputs: K (capital) and L (Labor), which have input prices *r* and *w* respectively. Furthermore, assume that the firm's price is normalized to 1, and that the production function is given by a constant elasticity of substitution (CES) production function:

$$y = F\left(K, L\right) = \left[K^{\rho} + L^{\rho}\right]^{\frac{1}{\rho}}$$

[***Quick aside:***

Just as an aside, and as a point of interest (since some of the papers we may read might use the CES function), here are some of the properties of the function above. As its name suggests, the elasticity of substitution σ (which in this case would measure the *curvature* of the production isoquant) is constant for the CES production function. More specifically, the elasticity of substitution measures the percentage change in the factor ratio divided by the percentage change in the *technical rate of substitution*, whilst holding output fixed. That is:

$$\sigma = \frac{\dfrac{\Delta\left(K/L\right)}{K/L}}{\dfrac{\Delta\left(dK/dL\right)}{dK/dL}} = \frac{d\ln\left(K/L\right)}{d\ln\left|dK/dL\right|} = \frac{d\ln\left(K/L\right)}{d\ln\left|TRS\right|}$$

The usefulness of the CES production function in replicating different types of production functions becomes apparent as we vary the parameter ρ . For example:

Case I: Linear production function:

When we set $\rho = 1$, the production function above becomes: $y = K + L$, where the two inputs, capital and labor are *perfect substitutes*.

Case 2: Cobb Douglas production function:

When ρ tends to 0, i.e. $\lim_{\rho \to 0} y$, the isoquants of the CES production function look very much like those of the Cobb-Douglas production function. This can be shown a variety of different ways mathematically, but the easiest is to compute the technical rate of substitution. As such, the two inputs in this case are *imperfect substitutes*, where the production isoquants are downward sloping.

Case 3: Leontieff Production function:

When ρ tends to $-\infty$, i.e. $\lim_{\rho \to -\infty} y$, the production isoquants become "L-shaped", which we associate with the *perfect complements* case for inputs.

-- End Aside]

a. [5 pts] Calculate the partial derivatives of y with respect to K and with respect to L.

Answer:

$$y = \left[K^{\rho} + L^{\rho}\right]^{\frac{1}{\rho}}$$

$$\frac{\partial F}{\partial K} = \frac{1}{\rho} z^{\frac{1}{\rho}-1} \rho K^{\rho-1} = z^{\frac{1-\rho}{\rho}} K^{\rho-1}, \text{ where } z = \left[K^{\rho} + L^{\rho}\right]$$

$$\frac{\partial F}{\partial L} = \frac{1}{\rho} z^{\frac{1}{\rho}-1} \rho L^{\rho-1} = z^{\frac{1-\rho}{\rho}} L^{\rho-1}$$

b. [5 pts] Consider that the total derivative of the production function above would be calculated as:

$dy = \frac{\partial F}{\partial K} dk + \frac{\partial F}{\partial L} dL$. The technical rate of substitution measures how one of the inputs must adjust in order to keep output constant (i.e. when $dy = 0$) when the other changes, and can be calculated from the total derivative above as: $TRS = \frac{dK}{dL} = -\frac{\partial F / \partial L}{\partial F / \partial K}$. Using your answer to part (a), calculate the technical rate of substitution for the CES production function above.

Answer:

$$TRS = \frac{dK}{dL} = -\frac{z^{\frac{1-\rho}{\rho}} L^{\rho-1}}{z^{\frac{1-\rho}{\rho}} K^{\rho-1}} = -\left(\frac{L}{K}\right)^{\rho-1} = -\left(\frac{K}{L}\right)^{1-\rho}$$

c. [4 pts] Using your answer from part (b), re-write the equation so that you have (K/L) as a function of the TRS. [Hint: take the absolute values of the TRS and then re-arrange for (K/L)]].

Answer:

$$|TRS| = \left(\frac{K}{L}\right)^{1-\rho}$$

$$\Rightarrow \left(\frac{K}{L}\right) = |TRS|^{\frac{1}{1-\rho}}$$

d. [4 pts] Take logs of both sides and differentiate to show that the elasticity of substitution is a constant

Answer:

$$\ln(K/L) = \frac{1}{1-\rho} \ln|TRS|$$

$$\Rightarrow d\ln(K/L) = \frac{1}{1-\rho} d\ln|TRS|$$

$$\Rightarrow \sigma = \frac{d\ln(K/L)}{d\ln|TRS|} = \frac{1}{1-\rho}$$

An alternative way to think about the producer's problem is to view it as a firm's choice to produce a certain amount of output, $\overline{y}$, and then they wish to minimize the amount of costs needed to produce that amount of output. This is typically the "Dual" problem to the firm's decisions.

e. [6 pts] Given input prices *r* and *w* for K and L respectively, write out the Lagrangean for the firm's cost minimization problem for producing an amount of output $\overline{y}$.

Answer: The firm's problem is to $\min_{K,L} rK + wL$ *subject to:* $\left[K^{\rho} + L^{\rho}\right]^{\frac{1}{\rho}} = \overline{y}$. *It is actually easier to transform the constraint a bit, so that the production function is now:* $K^{\rho} + L^{\rho} = \overline{y}^{\rho}$.

As such, the lagrangean for the problem is:

$$\mathcal{L} = rK + wL + \lambda\left(\bar{y}^{\rho} - K^{\rho} - L^{\rho}\right)$$

f. [6 pts] Derive the first order conditions (FOC) for the problem above.

Answer:

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda\rho K^{\rho-1} = 0 \qquad \text{(I)}$$

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda\rho L^{\rho-1} = 0 \qquad \text{(II)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = K^{\rho} + L^{\rho} = \bar{y}^{\rho} \qquad \text{(III)}$$

g. [10 pts] Solve for the optimal values of K^* and L^*.

Answer: Using equations (I) and (II) above, we can eliminate λ *to get the expression that the absolute value of the marginal rate of technical substitution equals the (input) price ratio:*

$$\left(\frac{K}{L}\right)^{\rho-1} = \frac{r}{w} \Rightarrow K^* = \left(\frac{r}{w}\right)^{\frac{1}{\rho-1}} L^* \qquad \text{(IV)}$$

We now have an expression for K in terms of L. Finally, by plugging (IV) into (III) (which we have not used up to this point), we can solve for L first and then K* as follows:*

$$\left(\frac{r}{w}\right)^{\frac{\rho}{\rho-1}} L^{\rho} + L^{\rho} = \bar{y}^{\rho}$$

$$\Rightarrow \left(L^*\right)^{\rho}\left[1 + \left(\frac{r}{w}\right)^{\frac{\rho}{\rho-1}}\right] = \bar{y}^{\rho}$$

$$\Rightarrow \left(L^*\right)^{\rho}\left[w^{-\frac{\rho}{\rho-1}}\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)\right] = \bar{y}^{\rho}$$

$$\Rightarrow L^* = w^{\frac{1}{\rho-1}}\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \bar{y}$$

$$\Rightarrow K^* = r^{\frac{1}{\rho-1}}\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \bar{y}$$

h. [10 pts] Calculate the “Value function” for the cost function, which is obtained by plugging in the optimal values of K* and L* that you obtained in part (g) above into the cost function itself.

Answer: By plugging in the optimal values for K and L* into the cost function, we can obtain total costs as a function of the input prices and the level of output the firm wishes to achieve. That is:*

$$
\begin{aligned}
C(r,w,\bar{y}) &= rK^* + wL^* \\
&= r^{1+\frac{1}{\rho-1}}\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}}\bar{y} + w^{1+\frac{1}{\rho-1}}\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}}\bar{y} \\
&= \bar{y}\left(r^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right)\left(w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \\
&= \bar{y}\left(r^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \\
&= \bar{y}\left(r^{\theta} + w^{\theta}\right)^{\frac{1}{\theta}}
\end{aligned}
$$

[Notice that the form of the cost function is the same as the original CES production function except with ρ replaced with θ instead, and the inputs replaced by their factor prices.]

<u>Question 2: IS-LM question</u> (50%)

Consider the following IS-LM model:

$$C = 200 + \frac{1}{4}(Y - T)$$

$$I = 150 + \frac{1}{4}Y - 1000r$$

$$G = 250$$

$$T = 200$$

$$L(r,Y) = 2Y - 8000r$$

$$\frac{M}{P} = 1600$$

a. [8 pts] Derive the IS equation

Answer: To get the IS equation, just add up C+I+G as follows:

$$IS: Y = 200 + \frac{1}{4}(Y - 200) + 150 + \frac{1}{4}Y - 1000r + 250$$

$$\Rightarrow Y = 550 + \frac{1}{2}Y - 1000r$$

$$\Rightarrow Y = 1100 - 2000r \qquad (1)$$

b. [8 pts] Derive the equation for the LM curve. [Hint, you want to write it as interest rates as a function of everything else].

Answer: To get the LM equation, just set demand equal to supply for real money balances:

$$\left(\frac{M}{P}\right) = L(r, Y)$$
$$\Rightarrow 1600 = 2Y - 8000r$$
$$\Rightarrow 8000r = -1600 + 2Y$$
$$\Rightarrow r = -0.2 + 0.00025Y \qquad (2)$$

c. [8 pts] Solve for the equilibrium output. [Hint: substitute the LM equation into the IS equation and then solve for output]

Answer:

$$Y = 1100 - 2000(-0.2 + 0.00025Y)$$
$$= 1500 - 0.5Y$$
$$\Rightarrow Y^* = 1000$$

d. [8pts] Solve for the equilibrium interest rate. [Hint: substitute the value you obtained for output in part (c) into either the IS or LM equations, and solve for interest rates. You should get the same number from both if you did the math correctly]

Answer: Plugging into the LM equation, we get:

$$r = -0.2 + 0.00025 \times 1000$$
$$\Rightarrow r^* = -0.2 + 0.25 = 0.05$$

Thus interest rates are 5% in equilibrium.

e. [8pts] Calculate the values of consumption and investment in equilibrium. Verify that the sum of C, I and G add up to output.

Answer:

$$C^* = 200 + 0.25 \times (1000 - 200) = 400$$
$$I^* = 150 + 0.25 \times 1000 - 1000 \times 0.05 = 350$$
$$G = 250$$
$$\Rightarrow C^* + I^* + G = 400 + 350 + 250 = 100$$

f. [10 pts] Suppose that the money supply increases to M/P = 1840. Solve for Y, r, C and I and describe in words the impact of a monetary expansion.

Answer: Intuitively, a monetary expansion should impact the LM curve. It should create a surplus of money in the money market and cause interest rates to go down, shifting the LM curve out.

The decrease in interest rates should spur investment and output and we would move along the IS curve in the short run. Since output goes up, consumption should too.

$$1840 = 2Y - 8000r$$
$$\Rightarrow r = -0.23 + 0.00025Y$$

This is the new LM curve. Plugging into the IS curve yields:

$$Y^* = 1100 - 2000(-0.23 + 0.00025Y^*)$$
$$\Rightarrow 1.5Y^* = 1560$$
$$\Rightarrow Y^* = 1040$$
$$\Rightarrow r^* = -0.23 + 0.00025 \times 1040 = 0.03$$
$$C^* = 200 + 0.25 \times (1040 - 200) = 410$$
$$I^* = 150 + 0.25 \times 1040 - 1000 \times 0.03 = 380$$

Thus our observations are verified.