Intermediate Macroeconomics

ECON 302

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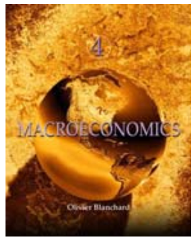
Lecture 14:

- Stylized Facts of Growth
- Solow Growth Model
- Golden Rule of Capital Accumulation
- Growth Accounting

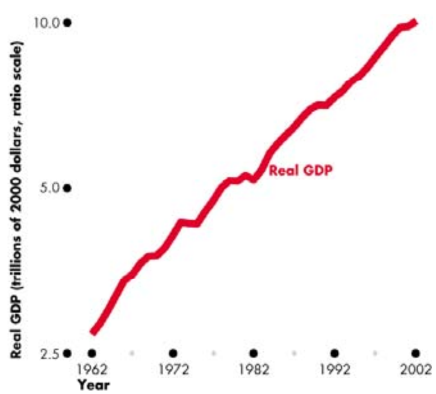


Key Concepts In This Lecture

- Stationary State
- Efficiency Units
- Balanced Growth Path or Steady State
- The Golden Rule of Capital Accumulation
- Sources of Growth
  - Growth Accounting
  - Solow Residual / Total Factor Productivity

What is Economic Growth

- Over time, real GDP is increasing
  - Production rising
  - Rising incomes
  - Improvements in standard of living
- Economic Growth rate is measured as the percentage change in real GDP



Heart of Economic Growth

- Question: What drives economic growth?
- In order to understand how to answer the question, we have to think about supply, and what drives production in the long run!

$$Y = F(K, L, A)$$

- Capital
- Labor
- Technology

Basics of the Solow Growth Model

- Supply side – Determines output given capital stock and the labor force
  - ignore technology and population growth for the moment
- Demand side – Determines consumption, savings, investment etc given output
- Long Run – **Key feature is that Investment adds to the Capital stock (K)**

Stationary Equilibrium

Definition: **Stationary Equilibrium** – Capital stock is no longer growing

Deriving the Stationary State

- Start with the idea that investment adds to the stock of capital and replenishes depreciated capital, i.e.:
  - $I = \Delta K + \delta K \quad 0 < \delta < 1 \quad (1)$
- Supply side: $Y = F(K,L)$ (2)
- Demand side: $I = S + T - G - NX$
- Assume that in the long run:
  - Government balances its budget, G = T
  - Trade is balanced, NX = 0

  } $I = S$ (3)

Deriving the Stationary State

- Assume further that savings is proportional to income, i.e. :
  - i.e. $S = sY, \quad 0 < s < 1$ (4)

    where $0<s<1$ is the marginal propensity to save
- Hence, (3) implies that $I = sY$.
- Substitute into (1) to obtain:
  - $$\Delta K = I - \delta K = sY - \delta K \quad (5)$$

The Stationary State

- Recall that a stationary state is one where the capital stock does not grow, i.e. $\Delta K = 0$
- Hence at a stationary state: $sY = \delta K$ (6)
- Incorporating the production function (2) into (6) yields:
  - $$Y = F(K,L) = \frac{\delta}{s}K \qquad (7)$$
- Note that (7) is a single equation in the stationary capital stock, i.e. $F(K,L)=(\delta/s)K$

Practice Question

- Suppose that the production function in (2) is a Cobb Douglas production function, i.e.
  - $$F(K,L) = AK^{\alpha}L^{1-\alpha}$$
- Calculate the stationary equilibrium level of the capital stock, $\bar{K}$.
  - Hint: Your solution for the capital stock at the stationary equilibrium should be a function of the exogenous value of labor, $\bar{L}$, as well as all the parameters, A, s, δ and α.
- Solution:

Output at the Stationary State

- Once we have obtained the stationary level of the capital stock, it is relatively straightforward to obtain the level of output at the stationary state.
- That is, given $\bar{K}$, and $\bar{L}$, stationary output is obtained by plugging these into the production function, equation (2).
- Illustration with Cobb Douglas Production Function:

$$\bar{Y} = A\bar{K}^{\alpha}\bar{L}^{1-\alpha} = A\left(\frac{As}{\delta}\right)^{\frac{\alpha}{1-\alpha}}\bar{L}^{\alpha}\bar{L}^{1-\alpha} = A^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}\bar{L}$$

Output at the Stationary State (cont.)

- Output at the stationary state:

$$\bar{Y} = A^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}\bar{L}$$

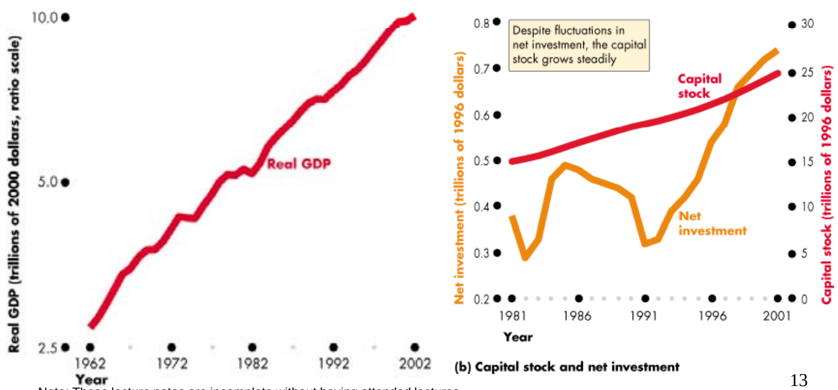
- Or, output per head (output per capita) can be written as:

$$\frac{\bar{Y}}{\bar{L}} = A^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

- Note:
  - Increases in the level of technology, and savings increase the level of output at the steady state
  - Both capital and output are constant at the steady state

In the Real World

- In the stationary state, Y and K are constant.
- However, facts show that they both grow on average at the same rate.



Stylized Facts of Growth (Kaldor, 1967)

- In our simple growth model, we have to account for the facts that Y and K grow over time.

- In addition, over a long period of time, we have observed the following "stylized facts of growth" (Kaldor, 1967)[1]:

1. Y/K is constant
2. K/L grows at a constant rate
3. w grows at a constant rate
4. r is constant

1. Kaldor, N. (1967) Strategic Factors in Economic Development, New York, Ithaca

Steady State (Balanced Growth Path)

- Define a ***Balanced Growth Path*** (also known as the ***steady state***) by:
  - A constant ratio of Y to K

- To address the issue that Y and K are not constant over time, first we transform the variables into per-capita terms using the CRS property of the production function

- Production function in equation (2):
  - Y = F (K, L)

- Recall that the production function is assumed to be Neoclassical, i.e. it exhibits constant returns to scale:
  - $\lambda Y = F(\lambda K, \lambda L) \qquad \lambda > 0$

Transforming the Variables

- Let lowercase letters represent per-capita values.

- Define: k = K/L ; y = Y/L

- Let $\lambda = \frac{1}{L}$ and multiply Y by λ

- Hence, $\frac{Y}{L} = F\left(\frac{K}{L},1\right)$, or in per capita terms:
  - $y = f(k);$ where $\left(k = \frac{K}{L}\right)$ (8)

- Properties of *f :*

$$f'(k) > 0;\ f''(k) < 0;\ \lim_{k\to 0} f'(k) = \infty;\ \lim_{k\to\infty} f'(k) = 0$$

Factor Prices in Per Capita Terms

- Recall from Lecture 13, that in the long run, factors are paid their marginal products, i.e.
  - MPK = r + δ (marginal product of capital = real rental rate of capital + rate of depreciation)
  - MPL = w (marginal product of labor = real wage)
- Question: Derive the expressions for r using the transformed variables y, and k.
- Solution: $r = \frac{\partial Y}{\partial K} - \delta.\ \ Since\, Y = yL = Lf(k),$

$$\frac{\partial Y}{\partial K} = L * f'(k)\frac{d\left(\frac{K}{L}\right)}{dK} = Lf'(k)\frac{1}{L} = f'(k)$$

Hence, $r = f'(k) - \delta$

Question: Real Wage

- Derive the expression for the real wage as a function of the transformed variables, y (= f(k)) and k.
- Solution:

Deriving The Steady State

- Investment per capita:

$$i = \frac{I}{L} = \frac{\Delta K + \delta K}{L} = \Delta k + \delta k \qquad (9)$$

- Since in equilibrium, I = sY, in per capita terms, this means:

$$\frac{I}{L} = \frac{sY}{L} \Rightarrow \Delta k + \delta k = sf(k)$$

- Hence, we obtain the equation for capital accumulation:

$$\Delta k = sf(k) - \delta k \qquad (10)$$

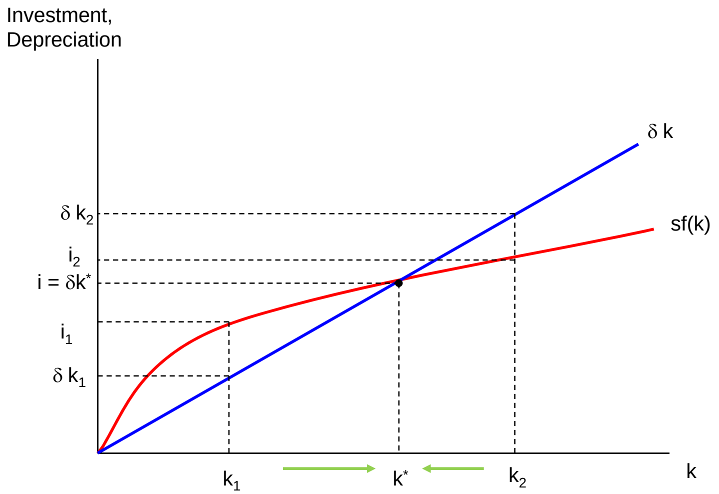
Dynamics… Capital Accumulation

Capital Accumulation:

$$\Delta k = sf(k) - \delta k$$

- Note:
  1. If $sf(k) = \delta k$, then we obtain a stationary state as before, since $\Delta k = 0$
  2. If $sf(k) > \delta k$, then $\Delta k > 0$, i.e. the stock of capital in the economy will rise
  3. If $sf(k) < \delta k$, then $\Delta k < 0$, i.e. the stock of capital in the economy will fall

Dynamics in Solow Growth Model



Implications of Dynamics

Note the following:

- Stationary State is stable, since if it exists, *sf(k)* line must cut δk line from above.
  - ➢ Why?
- In this model, growth takes place only during an adjustment period, e.g. in response to:
  1. Destruction of some capital stock
  2. Increase in the saving rate

Question…

- Can the growth we observe in the world be the result of very slow adjustment to a stationary state?
- Answer:
  - ➢
- Consider two other sources of steady-state growth:

$$Y = F\left(K, L, A\right)$$

- ➢ Population growth
- ➢ Technological Progress

Incorporating Population Growth

- Let the population, L, grow at a constant rate, n.

$$\frac{\Delta L}{L} = n \quad \text{or equivalently} \quad \frac{L'}{L} = n$$

- Recall from equation (8), *y = f(k),* and that in the steady date:

$$\frac{y}{k} = \text{constant}$$

$$\text{i.e. } \frac{\Delta y}{y} = \frac{\Delta k}{k} \qquad (11)$$

Incorporating Population Growth (cont.)

- Differentiate the per-capita production function (8):

$$\frac{\Delta y}{y}=\frac{kf'(k)}{f(k)}\frac{\Delta k}{k} \qquad (12)$$

- How can we reconcile equations (11) and (12)?

- Since $f'(k)>0$ and $f''(k)<0$, equations (11) and (12) imply that at a steady state:

$$\frac{\Delta y}{y}=\frac{\Delta k}{k}=0$$

i.e. y and k are both constant (in per capita terms)

Issues…

- Without technological progress, output per head and capital per head have to be constant – already contradicts one of the stylized facts of growth.
  - #2: K/L grows at a constant rate

- To find condition which ensures constancy, recall: k=K/L
  - K grows by I – δK
  - L grows at n

∴ For k to be constant, K needs to grow at the proportional rate n.

Implied Growth Rate for Capital

- For k to be constant, K needs to grow at the proportional rate n, i.e.:

$$\frac{\Delta k}{k}=0\Rightarrow\frac{\Delta K}{K}=\frac{\Delta L}{L}=n$$

- Incorporating equation (5) into above yields:

$$\frac{I-\delta K}{K}=n$$

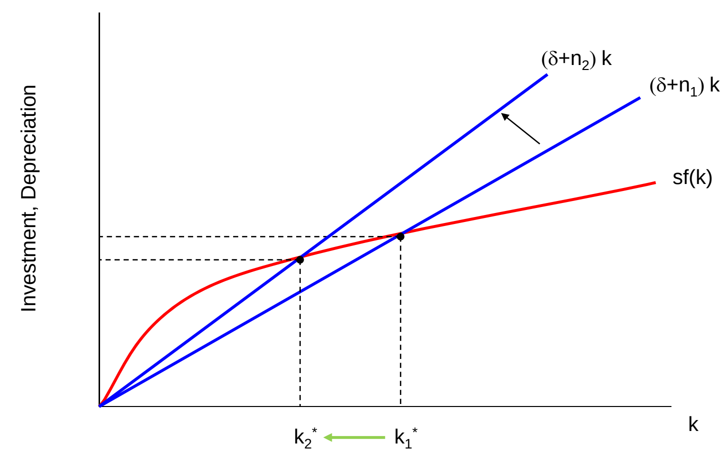
or $i=(n+\delta)k$

$$\Rightarrow \qquad sf(k)=(n+\delta)k$$

- This says: Gross investment must equal break even investment at the steady state.

Increase In Population

Suppose n increases from n_1 to n_2:



Implication for Factor Prices

What does the model imply about factor prices?

- Recall that $r = f'(k) - \delta$ and $w = f(k) - k.f'(k)$
- Since k is constant in equilibrium, both r and w are constant.
- This contradicts one of the other stylized facts of growth
  - #3: w should grow

Incorporating Technological Progress

How to Include Technology?

- Data shows that both y and k grow
- Exogenous technological change: Y = F(K,L,A) where A is "Technology parameter" which grows exogenously
- Consider 3 possible ways technology can enter:
  - Y=AF(K,L) (Hicks Neutral)
  - Y=F(AK,L) (Solow Neutral)
  - Y=F(K,AL) (Harrod Neutral)

Labor Augmenting Technology

- Define: AL = "efficiency units" of labor
- Here A can be thought of as education, training etc, "human capital"
- For the Cobb-Douglas production function, all three forms are equivalent, but in general, this may not be the case (… show how this is true for the Cobb Douglas production function!)
- Solow model deals with "**labor-augmenting**" or **Harrod Neutral** form of technological progress

Harrod Neutral Technological Progress

- We can incorporate technology using the previous strategy:
  - Transform output and other variables into efficiency units using the CRS property of the production function!
- Denote efficiency units with "~", i.e.: $\tilde{k} = \frac{K}{AL}; \tilde{y} = \frac{Y}{AL}; etc$
- Let $\lambda = \frac{1}{AL}$ and multiply Y by λ
- Hence, $\frac{Y}{AL} = F\left(\frac{K}{AL}, 1\right)$, or in per capita terms:
  - $\tilde{y} = f\left(\tilde{k}\right);$ (13)

Steady State Growth

- At the steady state:

$$\frac{\Delta\tilde{y}}{\tilde{y}} = \frac{\Delta\tilde{k}}{\tilde{k}} = 0$$

- Assume that A grows at the rate g
- Since $\tilde{k} = \frac{K}{AL}$,

$$\frac{\Delta\tilde{k}}{\tilde{k}} = \frac{\Delta K}{K} - \left(\frac{\Delta A}{A} + \frac{\Delta L}{L}\right) \Rightarrow \frac{\Delta K}{K} = \frac{\Delta A}{A} + \frac{\Delta L}{L}$$

- i.e. the growth rate of K must equal the sum of the growth rates of A and L

Growth Rates of Key Variables

- Rate of growth of K: $\frac{\Delta K}{K} = \frac{sF(K,AL) - \delta K}{K}$
- Rate of growth of A and L: $\frac{\Delta A}{A} = g;\ \frac{\Delta L}{L} = n$
- Thus at the steady state where $\Delta\tilde{k} = 0$

$$\frac{sF(K,AL) - \delta K}{K} = n + g$$

$$\Rightarrow sF(K,AL) - \delta K = (n+g)K$$

- Divide both sides through by AL to get:

$$sf(\tilde{k}) = (n + g + \delta)\tilde{k} \qquad (14)$$

Capital Accumulation with Technological Progress and Population Growth

- Thus the equation for capital accumulation is given by:

$$\Delta\tilde{k} = sf(\tilde{k}) - (n + g + \delta)\tilde{k} \qquad (15)$$

- As before, note that:
  1. At the steady state, $\Delta\tilde{k} = 0, \Rightarrow sf(\tilde{k}) = (n + g + \delta)k$
  2. If $sf(\tilde{k}) > (n + g + \delta)\tilde{k},\ \Delta\tilde{k} > 0$, i.e. the stock of capital (in terms of efficiency units) in the economy will rise
  3. If $sf(\tilde{k}) < (n + g + \delta)\tilde{k}, \Delta\tilde{k} < 0$, i.e. the stock of capital (in efficiency units) will fall in the economy

Summary of Growth Rates

	Variable	Growth Rate
Levels:	K	n+g
	L	n
	A	g
	Y	n+g
Per Capita:	K/L	g
	Y/L	g
Efficiency Units:	$\tilde{k}$	0
	$\tilde{y}$	0

Behavior of Real Wage and the Real Interest Rate

- Question: As earlier, derive the expression for the real wage as a function of the transformed variables, $\tilde{y} = f\left(\tilde{k}\right)$ and $\tilde{k}$
  - Hint: As before, *r = MPK – δ; w = MPL!*

- Solution:

What we have managed to explain:

We have now explained the four stylized facts presented earlier:

1. Y/K is constant
2. K/L grows at a constant rate – g
3. r is constant
4. w grows at a constant rate – g

- Problem: Rate of growth, g, is exogenous
  - Models exist which endogenize this growth rate – Endogenous Growth models!

Golden Rule of Capital Accumulation

- The Solow growth model gives the following steady state equation

$$sf\left(\tilde{k}\right) = \left(n + g + \delta\right)\tilde{k} \qquad (14)$$

where s,n,g,δ are exogenous constants

- Equation (14) is one equation in one unknown:- $\tilde{k}$

Question:

Is the value of capital obtained from (14) the one which maximizes social welfare?

Maximizing Social Welfare

- Social Welfare: Adopt a simple rule – consumption per capita in the steady state.

- Trade off: More consumption now lowers savings and thus lowers investment. This implies that there is less capital and less consumption in the future.

- *Golden Rule* gives the level of capital that maximizes consumption on the steady state balanced growth path.

Maximizing Consumption at the Steady State

- Define consumption as the difference between output and savings, i.e.:

$$\tilde{c} = f(\tilde{k}) - sf(\tilde{k}) \qquad (16)$$

- At the steady state, $\Delta\tilde{k} = 0, \Rightarrow sf\left(\tilde{k}\right) = \left(n+g+\delta\right)k$

- Thus, plugging into (16) yields:

$$\tilde{c}^* = f\left(\tilde{k}^*\right) - \left(n+g+\delta\right)\tilde{k}^* \qquad (17)$$

- Objective: Find the value of $\tilde{k}$ that maximizes $\tilde{c}^*$

Deriving the Golden Rule

- The value of $\tilde{k}$ that maximizes $\tilde{c}^*$, satisfies:

$$f'\left(\tilde{k}_{GR}\right) = n+g+\delta \qquad \text{i.e. } MPK = n + g + \delta$$

- In a competitive economy, the firm maximizes profits:

$$F(K,AL) - \delta K - rK - wL$$

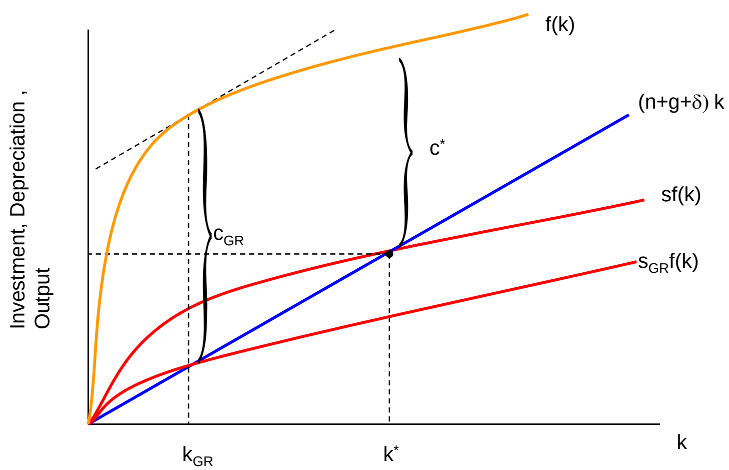
- It chooses K to satisfy: MPK = δ + r

- Thus, the Golden Rule for a *competitive* economy is:

$$r = n + g$$

Real Rate of Interest = Real Rate of Growth

The Golden Rule



Sources of Growth – Growth Accounting

From the Production Function, Y = F(K,L,A), output can grow for one of three reasons:

1. Growth in K
2. Growth in L
3. Growth in A (Technology).

- Decomposing Growth in Y amongst these three sources of growth is known as **growth accounting**.

- Part of growth due to A is called **Total Factor Productivity**.

- Illustration: Cobb Douglas Production Function

$Y = AF(K,L) = AK^{\alpha}L^{1-\alpha}$ $\quad 0 < \alpha < 1$

$\therefore \; \Delta Y = \Delta A K^{\alpha}L^{1-\alpha} + \alpha \Delta K A K^{\alpha-1}L^{1-\alpha} + (1-\alpha)\Delta L A K^{\alpha}L^{-\alpha}$

Dividing both sides by Y:

$$\therefore \quad \frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \alpha\frac{\Delta K}{K} + (1-\alpha)\frac{\Delta L}{L}$$

We have data on: $\frac{\Delta Y}{Y}, \alpha, \frac{\Delta K}{K}, \frac{\Delta L}{L}$

$\therefore$ we can calculate $\frac{\Delta A}{A}$ from above.

By definition, it is the "unaccounted" or "residual" part of growth:

$$\frac{\Delta A}{A} = \frac{\Delta Y}{Y} - \left(\alpha\frac{\Delta K}{K} + (1-\alpha)\frac{\Delta L}{L}\right)$$

- For this reason, it is also known as the **Solow Residual**.
- In the UK, about half of growth is attributed to TFP growth.
- In the US, there is a more even split between the three sources.

Summary

1. Solow Growth Model attributed growth to
   - Capital Accumulation
   - Population Growth
   - Technological Progress
2. Of all of these, capital accumulation and population growth by themselves are unable to account for Kaldor's stylized facts of growth
3. Stylized facts of growth are explained through exogenous technological progress

Summary

4. Golden Rule of Capital Accumulation determines the level of capital that maximizes consumption on the balanced growth path
   - In a competitive economy, such a rule sets equates the real interest rate to the real growth rate of the economy (n+g)
5. Total Factor Productivity (TFP) is the part of growth that we cannot account for (from capital accumulation and population growth), and thus attribute towards technological progress.