Calculus and Statistics Review Notes

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This math review sheet was written to provide you a with a quick reference for some of the calculus and statistics concepts you will encounter within the course. Please consult a mathematical textbook for a more detailed treatment on any particular subject area or materials. A good text related to Economics is "Mathematics for Economists" by Carl P. Simon and Lawrence Blume (Norton, 1994).

Calculus

Def 1: A function f is linear if its slope is a constant and does not depend on any variable. In addition, a function is linear if its graph is a straight line. Linear functions (first order polynomials) of this form are typically written as:

$$y = f\left(x\right) = mx + c$$

where m is the slope, and c is the intercept.

Def 2: Let f be a function that is defined (and real valued) on the interval [a,b]. For any $x \in [a, b]$, form the quotient

$$\phi\left(t\right) = \frac{f(t) - f(x)}{t - x} \qquad (a < t < b, t \neq x)$$

and define $f'(x) = \lim_{t \to x} \phi\left(t\right)$, provided that this limit exist. This is called the **derivative of f**. If f' is defined at a point x, we say that **f is differentiable at x**. If f' is defined at every point on the set $E \subset [a, b]$, we say that **f is differentiable on E**. The derivative of $y = f\left(x\right)$ is denoted $f'\left(x\right)$ or $\frac{dy}{dx}$ or $\frac{df}{dx}$.

Def 3: The slope of a function (whether linear or nonlinear) is obtained by calculating its derivative, i.e. $f'(x) = \frac{\Delta y}{\Delta x}$, where Δ represents "a change in". If this change is very, very small, e.g. as in definition 2, $t \to x$, then we replace the Δ with d, and write $f'\left(x\right) = \frac{dy}{dx}$. Note that for the linear function case, i.e. $y = mx + c$, $\frac{dy}{dx} = m$, a constant. Hence linear functions have a constant slope which does not depend on x (as stated above in definition 1).

Application:
To **differentiate** a function, $y = f(x)$, we can apply the following set of rules. Suppose that k, A, c, n are arbitrary constants, and that g is a differentiable function of x. Then, the derivative (with respect to x) of:

1. $\frac{d}{dx}(k) = 0$

2. $\frac{d}{dx}(x^k) = kx^{k-1}$

3. $\frac{d}{dx}\left(Ax^k\right) = Akx^{k-1}$

4. $\frac{d}{dx}\left(Ax^k + c\right) = Akx^{k-1}$ [This uses the additive property of differntiation - i.e. that the derivative of a sum, is the sum of the derivatives!]

5. $\frac{d}{dx}\left(Ax^k + Bx^n\right) = Akx^{k-1} + Bnx^{n-1}$ [As above!]

6. $\frac{d}{dx}\left(g(x)^n\right) = n\left(g(x)^{n-1}\right).g'(x)$, [Note: This is an application of the chain rule!]

Each of these derivatives (on the right hand side of the equality) are also referred to by $f'(x)$. The table below shows you the solution if you differentiate $f(x)$:

$f(x)$	$f'(x)$
x^m	mx^{m-1}
Ae^{bx}	Abe^{bx}
$\ln x$	$\frac{1}{x}$
a^x	$a^x \ln a$

For functions of x, the following three rules are useful. Let u, v be differentiable functions of x, i.e. $u = g(x)$ and $v = h(x)$. In addition, let f be a (composite) function of u, i.e.

- **Product Rule**: Suppose $y = uv$. Then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.
- **Quotient Rule**: Suppose $y = \frac{u}{v}$. Then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

In addition, if we let y be a (composite) function of u, i.e. $y = f(u)$ where $u = g(x)$, then

- **Chain Rule**: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Notes:
For a function $y = f(x)$. If the first derivative (or **first-order condition**)

1. $f'(x) > 0$, then the function y is increasing
2. $f'(x) < 0$, then the function y is decreasing
3. $f'(x) = 0$, then we have a **stationary point or critical point** of f, i.e. the slope is zero (- for a single variable function, the graph is flat!).

For #3 above, this can be used to show if we have a maximum or minimum. To do so, we need to look at the **second order condition** by examining the **second derivative**. The second derivative is a higher order derivative, and is obtained by differentiating the first derivative function. Similarly, we can obtain third (and higher order) derivatives by repeatedly differentiating (provided of course that those functions, and the derivatives exist). Thus the following rules can be applied:

If x_0 is a critical point of a function f, then:

1. If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a **max** of f
2. If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a **min** of f
3. If $f'(x_0) = 0$ and $f''(x_0) = 0$, then x_0 can be a max, a min or neither (a point of inflexion).

Def 4: If f is a function of several variables, to calculate the **partial derivative** with respect to a certain variable, treat the remaining variables as constants and differentiate as usual by using the rules of one-variable calculus. If $z = f(x, y)$ is a function of two variables, the two partial derivatives are denoted $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, or $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, or f_x and f_y.

Just as you can calculate higher order deriviatives, it is possible to calculate higher order partial derivatives. Hence, it is possible to calculate $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and the cross partials $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ by differentiating $\frac{\partial f}{\partial x}$ by x, $\frac{\partial f}{\partial y}$ by y, and either $\frac{\partial f}{\partial x}$ by y or $\frac{\partial f}{\partial y}$ by x respectively.

The **Total Derivative**. Suppose we are interested in the behavior of a function $F(x, y)$ of two variables in the neighbourhood of a given point, (x^*, y^*). If we were to only look at the variation in the 'x-direction', i.e. by holding y fixed at y^* and examining the change from x^* to $x^* + \Delta x$, then we would calculate

$$F(x^* + \Delta x, y^*) - F(x^*, y^*) \approx \frac{\partial F}{\partial x}(x^*, y^*)\,\Delta x$$

If we were instead to hold x fixed at x^* and change y^* to $y^* + \Delta y$, we could calculate

$$F(x^*, y^* + \Delta y) - F(x^*, y^*) \approx \frac{\partial F}{\partial y}(x^*, y^*)\,\Delta y$$

If we allow both x and y to vary simultaneously, we can obtain an approximation to the **Total Derivative** by summing the effects of the one-variable changes:

Def 5: The Total Derivative for a function $F(x, y)$ in the neighbourhood of a given point (x^*, y^*) is approximately given by:

$$
\begin{aligned}
F\left(x^{*}+\Delta x, y^{*}+\Delta y\right)-F\left(x^{*}, y^{*}\right) &\approx \frac{\partial F}{\partial x}\left(x^{*}, y^{*}\right) \Delta x+\frac{\partial F}{\partial y}\left(x^{*}, y^{*}\right) \Delta y \\
\text{i.e. } dF &\approx F_{x} \Delta x+F_{y} \Delta y
\end{aligned}
$$

Statistics

You have probably seen all of these concepts before:

1. **Sample Mean**: $\bar{x}=\frac{1}{n} \sum_{i=1}^{n} x_{i}$
2. **Sample Variance**: $s^{2}=\frac{1}{n} \sum_{i=1}^{n}\left(x_{i}-\bar{x}\right)^{2}$
3. **Sample Covariance** (for a sample of n pairs of observations between variables x_i and y_i): $s_{xy}=\frac{1}{n} \sum_{i=1}^{n}\left(x_{i}-\bar{x}\right)\left(y_{i}-\bar{y}\right)=\frac{1}{n} \sum_{i=1}^{n} x_{i} y_{i}-\bar{x} \bar{y}$
4. **Correlation**:

$$
r=\frac{s_{xy}}{s_{x} s_{y}}=\frac{\sum\left(x_{i}-\bar{x}\right)\left(y_{i}-\bar{y}\right)}{\sqrt{\left[\sum\left(x_{i}-\bar{x}\right)^{2} \sum\left(y_{i}-\bar{y}\right)^{2}\right]}}
$$

Suppose that X is a random variable talking values x_i in a discrete distribution with $P\left(X=x_{i}\right)=p_{i}$. Then the concepts above can also be written as:

1. **Expectation (Mean)**: $\mu=E(X)=\sum x_{i} p_{i}$
2. **Variance**: $\sigma_{x}^{2}=Var(X)=E[(X-\mu)^{2}]=E\left(X^{2}\right)-\mu^{2}=\sum_{i=1}^{n}\left(x_{i}-\mu\right)^{2} p_{i}=\sum_{i=1}^{n} x_{i}^{2} p_{i}-\mu^{2}$
3. **Covariance** (for a sample of n pairs of observations between variables x_i and y_i): $\sigma_{xy}=Cov(X, Y)=E[(X-E(X))(Y-E(Y))]=E(XY)-E(X) E(Y)$
4. **Correlation**: $\rho=\frac{\sigma_{xy}}{\sigma_{x} \sigma_{y}}$